A Note on the Open Induction Principle

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Introduction

In the reference [2], I described a possible point-free interpretation of a new induction principle, the open induction principle, that was use by J.C. Raoult for a new presentation of Nash-Williams proof of Kruskal theorem [6]. In the point-free version, one uses generalised inductive definition twice iterated (system ID_2 , see [1]). When formulated in term of choice sequences, this reasoning needs the so called extended bar-induction [7], which is not derivable in the system of Kleene-Vesley [3].

Next question: the open induction can be formulated naturally also for reals and on the Cantor space. We show in this note that these versions can be proved in the system of Kleene-Vesley. Interestingly, bar induction is used in this justification, and it seems necessary.

1 Open Induction on Cantor's Space

Cantor space is the space of all infinite sequence of 0 and 1. We denote by $\alpha, \beta, \dots$ points of this space and we consider on this space the lexicographic ordering $<$. It is not difficult to find infinite descreasing sequence $1000\ldots > 0100\ldots > 00100\ldots$ and thus this ordering is not well-founded. This implies that the well-founded induction principle

$$\frac{(\forall\alpha)[[\forall\beta < \alpha\ P(\beta)]\to P(\alpha)]}{(\forall\alpha)P(\alpha)}$$

does not hold.

The following principle expresses however that the well-founded induction principle is valid if one considers only open properties. This means that if we have an open subset U of the Cantor space such that

$$(\forall\alpha)[[\forall\beta < \alpha\ \beta \in U]\to\alpha \in U]$$

then

$$(\forall\alpha)\alpha \in U.$$

We give a proof of this principle in intuitionistic mathematics using the bar theorem.

2 Proof of the Open Induction Principle

We can suppose that the open U is given by a decidable monotone property G on finite binary words. Furthemore, we say that k is a witness for the fact that G bars σ iff G holds on any extension of σ of the form $\sigma b_1 \dots b_k$. Finally, we write $G|\sigma$ for "G bars σ."

We define when a finite sequence $\sigma = n_1 \dots n_k$ of natural numbers is *acceptable* (this will be a decidable property) and we associate to it a finite binary sequence $f(\sigma)$ of the same length. The empty sequence is acceptable, and $\sigma 0$ is acceptable iff σ is acceptable. Furthermore $f(\sigma 0) = f(\sigma)0$. For computing $f(\sigma(n+1))$ there are two cases. We define $\sigma(n+1)$ to be acceptable iff σ is acceptable and n is a witness that G bars $f(\sigma)0$. If $\sigma(n+1)$ is acceptable then $f(\sigma(n+1)) = f(\sigma)1$. If $\sigma(n+1)$ is not acceptable then $f(\sigma(n+1)) = f(\sigma)0$.

We can extend f naturally on infinite sequence. We still write $f(n_1 n_2 \ldots)$ for the infinite binary sequence that has all $f(n_1 \ldots n_k)$ as initial segment.

Lemma: For any sequence $n_1 n_2 \ldots$ there exists k such that $G(f(n_1 \ldots n_k))$.

Proof: By construction of f we have $\beta \in U$ whenever $\beta < f(n_1 n_2 \ldots)$. Hence we have $f(n_1 n_2 \ldots) \in U$.

Let $P(\sigma)$ mean that $G(f(\sigma))$ We prove now by bar induction that if $P|\sigma$ then $G|f(\sigma)$ if σ is acceptable. This holds if $P(\sigma)$. If it holds that for all n we have $G|f(\sigma n)$ if σn is accepatble, and if σ is acceptable then we have $G|f(\sigma)0$. Thus we have n a witness for the fact that G bars $f(\sigma)$. Then $\sigma(n+1)$ is acceptable and $f(\sigma(n+1)) = f(\sigma)1$. Hence $G|f(\sigma)1$. Since both $G|f(\sigma)0$ and $G|f(\sigma)1$, it follows that $G|f(\sigma)$.

3 Other example

The same argument will apply for the real interval $[0, 1]$ with the usual ordering of reals

$$\frac{(\forall \alpha \in [0,1])[[\forall \beta < \alpha \ \beta \in U] \rightarrow \alpha \in U]}{(\forall \alpha \in [0,1]) \alpha \in U.}$$

This principle is used for instance in the book of Lorenzen [4]. It can be used to prove the Heine-Borel property and the fact that $[0, 1]$ is connected. The same principle holds over the Baire space. However, it seems then to require two level of iterated inductive definitions, or alternatively the principle of extended bar induction.

Thus, we have three form of the open induction principle

1. on Cantor space,
2. on the real interval $[0, 1]$,
3. on Baire space.

To prove it on Cantor space and $[0, 1]$, we need the ordinary form of bar induction. To prove it on Baire space, we need an extended form of bar induction.

References

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