Intuitionistic Type Theory
Lecture 2

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Summer School on Types, Sets and Constructions
Hausdorff Research Institute for Mathematics
Bonn, 3 - 9 May, 2018

Agda

```
module N-rules where

data ℕ : Set where
  O : ℕ
  s : ℕ → ℕ

R : {C : ℕ → Set}
  → C O
  → ((n : ℕ) → C n → C (s n))
  → (c : ℕ) → C c
R d e O     = d
R d e (s n) = e n (R d e n)
```

Intuitionistic Type Theory 1972 - the "Spartan" version

Propositions as types:

$$
\begin{aligned}
\bot &= \emptyset \\
\top &= 1 \\
A \vee B &= A + B \\
A \wedge B &= A \times B \\
A \supset B &= A \rightarrow B \\
\exists x : A.B &= \Sigma x : A.B \\
\forall x : A.B &= \Pi x : A.B
\end{aligned}
$$

and

N	*the type of natural numbers*
U	*the type of small types - the universe*

1973-84

1973 (1975) *An Intuitionistic Theory of Types: Predicative part*. Adds a sequence of universes $U_0, U_1, U_2, \ldots$ and a general identity type $I(A, a, a')$. Weak combinatory version of the theory. Proof of normalization "by evaluation".

1979 (1982) *Constructive Mathematics and Computer Programming*. Adds W-type. Adds identity reflection and uniqueness of identity proofs ("extensional type theory"). Typed equality judgments. Meaning explanations.

1980 (1984) *Intuitionistic Type Theory* (Padova lecture notes by Sambin, Bibliopolis). Meaning explanations and justification of the rules. Universe ła Tarski.

The four forms of judgments

- $\Gamma \vdash A$ meaning A is a well-formed type,
- $\Gamma \vdash a : A$ meaning a has type A (the main judgment),
- $\Gamma \vdash A = A'$ meaning A and A' are equal types,
- $\Gamma \vdash a = a' : A$ meaning a and a' are equal elements of type A. (*Typed equality judgments*. Martin-Löf 1972 (and 1975) had untyped conversion $a = a'$.)

Distinction *proposition* vs *judgment*! (Cf Zeno's paradox of logic.)
$A = A'$ and $a = a' : A$ are called *definitional equalities*.

General inference rules

Rules which come before any rules for type formers:

- assumption rules
- substitution rules
- context formation rules
- equalities are equivalence relations

Of particular interest is the *rule of type equality* which is crucial for computation in types:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash A = B}{\Gamma \vdash a : B}$$

Rules for Π

Π-formation

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B}{\Gamma \vdash \Pi x : A.B}$$

Π-introduction

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : \Pi x : A.B}$$

Π-elimination

$$\frac{\Gamma \vdash f : \Pi x : A.B \qquad \Gamma \vdash a : A}{\Gamma \vdash f\, a : B[x := a]}$$

Π-equality (β and η)

$$\frac{\Gamma, x : A \vdash b : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x.b)\, a = b[x := a] : B[x := a]}$$

$$\frac{\Gamma \vdash f : \Pi x : A.B}{\Gamma \vdash \lambda x.f\, x = f : \Pi x : A.B}$$

(η is not in Martin-Löf 1972.)

Rules for Π

Preservation of equality

$$\frac{\Gamma \vdash A = A' \qquad \Gamma, x : A \vdash B = B'}{\Gamma \vdash \Pi x : A.B = \Pi x : A'.B'}$$

etc

Some predicative "full-scale" theories

- Intuitionistic Type Theory
- Constructive Set Theory, CST/CZF (Myhill, Aczel)
- Logical theory of constructions modelled by *Frege structures* (Aczel). Intuitionistic predicate logic + untyped lambda calculus with conversion + inductive definitions. Cf also Feferman's explicit mathematics.

Meaning explanations

- In Martin-Löf 1972 and 1973/75 there are *proofs of normalization*.
- In Martin-Löf 1979/82 "Constructive Mathematics and Computer Programming" there are *"meaning explanations"*, p 16

  In explaining what a judgment of one of the above four forms means, I shall limit myself to assumption free judgments. Once it has been explained what meanings they carry, the explanations can readily be extended so as to cover hypothetical judgments as well.

- Meaning explanations are also referred to as, *direct semantics, intuitive semantics, standard semantics, syntactico-semantical*.

Expressions and canonical forms for the Π, N, U-fragment

Expressions:

$$\begin{aligned} a \quad ::= \quad & 0 \mid s(a) \mid \lambda x.a \mid N \mid \Pi x : a.a \mid U \\ & \mid R(a, a, xx.a) \mid a\,a \end{aligned}$$

Canonical expressions:

$$v ::= 0 \mid s(a) \mid \lambda x.a \mid N \mid \Pi x : a.a \mid U$$

Computation rules

$a \Rightarrow v$ means a has canonical form v. We have

$$v \Rightarrow v$$

and

$$\frac{f \Rightarrow \lambda x.b \quad b[x := a] \Rightarrow v}{f\,a \Rightarrow v}$$

$$\frac{c \Rightarrow 0 \quad d \Rightarrow v}{R(c,d,xy.e) \Rightarrow v} \qquad \frac{c \Rightarrow s(a) \quad e[x := d, y := R(a,d,xy.e)] \Rightarrow v}{R(c,d,xy.e) \Rightarrow v}$$

The meaning of A type

The meaning of the categorical judgment $\vdash A$ is that A has a canonical type as value. In our fragment this means that either of the following holds:

- $A \Rightarrow N$,
- $A \Rightarrow U$,
- $A \Rightarrow \Pi x : B.C$ and furthermore that $\vdash B$ and $x : B \vdash C$.

The meaning of $A = A'$

The meaning of the categorical judgment $\vdash A = A'$ is that A and A' have equal canonical types as values. In our fragment this means that either of the following holds:

- $A \Rightarrow N$ and $A' \Rightarrow N$,
- $A \Rightarrow U$ and $A' \Rightarrow U$,
- $A \Rightarrow \Pi x : B.C$ and $A' \Rightarrow \Pi x : B'.C'$ and furthermore that $\vdash B = B'$ and $x : B \vdash C = C'$.

Remark: Martin-Löf 1982 says

> *Two canonical types A and B are equal if a canonical object of type A is also a canonical object of type B and, moreover, equal canonical objects of type A are also equal canonical objects of type B.*

The meaning of $a : A$

The meaning of the categorical judgment $\vdash a : A$ is that a has a canonical expression of the canonical type denoted by A as value:

- $A \Rightarrow N$ and either $a \Rightarrow 0$ or $a \Rightarrow s(b)$ and $\vdash b : N$,
- $A \Rightarrow U$ and either $a \Rightarrow N$ or $a \Rightarrow \Pi x : b.c$ where furthermore $\vdash b : U$ and $x : b \vdash c : U$,
- $A \Rightarrow \Pi x : B.C$ and $a \Rightarrow \lambda x.c$ and $x : B \vdash c : C$.

Assume we also have the type $\emptyset$, there would be no clause

- $A \Rightarrow \emptyset$

Hence we cannot have $a : \emptyset$, that is, *"simple minded consistency"*.

Hypothetical judgments - meaning explanations

Martin-Löf 1982:

- $y : B \vdash C$ *type* means that if $b : B$ then $C[y := b]$ *type*
- $y : B \vdash C = C'$ means that if $b : B$ then $C[y := b] = C'[y := b]$
- $y : B \vdash c : C$ means that if $b : B$ then $c[y := b] : C[y := b]$
- $y : B \vdash c = c' : C$ means that if $b : B$ then $c[y := b] = c'[y := b] : C[y := b]$

Meaning explanations

- Hypothetical judgments in general can have several assumptions.
- It remains to make sure that all the inference rules preserve the meaning of the judgments. See "Intuitionistic Type Theory" (Martin-Löf, Bibliopolis 1984).
- Are the meaning explanations vacuous?
- Tacit assumption of *well-foundedness*.

How to understand meaning explanations?

Introductory remarks (Martin-Löf, Bibliopolis 1984):

> *Mathematical logic and the relation between logic and mathematics have been interpreted in at least three different ways:*
>
> *(1) mathematical logic as symbolic logic, or logic using mathematical symbolism;*
>
> *(2) mathematical logic as foundations (or philosophy) of mathematics;*
>
> *(3) mathematical logic as logic studied by mathematical methods, as a branch of mathematics.*
>
> *We shall here mainly be interested in mathematical logic in the second sense. What we shall do is also mathematical logic in the first sense, but certainly not in the third.*

How to understand meaning explanations?

"pre-mathematically": mathematical logic as foundations (or philosophy) of mathematics;

metamathematically: mathematical logic as logic studied by mathematical methods, as a branch of mathematics.

The BHK-explanation

The justification of intuitionistic logic:

- Brouwer's analysis
- Heyting's calculus of intended constructions
- Kolmogorov's calculus of problems and solutions

This is "premathematics", e g Van Atten 2017 (SEP, article about the *Development of Intuitionistic Logic*)

> *For the moment, we note that the BHK-Interpretation or Proof Interpretation is not an interpretation in this mathematical sense, but is rather a meaning explanation.*

BHK and Martin-Löf

Martin-Löf's meaning explanations

- make BHK more *precise*, by introducing proof objects, canonical proofs, and formal computation rules
- make BHK more *general*, by explaining the meaning not only of constructive proofs but also of constructive mathematical objects

Both are "pre-mathematical".

Metamathematics

Metamathematics is the study of mathematics itself using mathematical methods. This study produces metatheories, which are mathematical theories about other mathematical theories. Emphasis on metamathematics (and perhaps the creation of the term itself) owes itself to David Hilbert's attempt to secure the foundations of mathematics in the early part of the 20th century. Metamathematics provides "a rigorous mathematical technique for investigating a great variety of foundation problems for mathematics and logic" (Kleene 1952, p. 59).

Metamathematical account of meaning explanations

- If we interpret the meaning explanation rules as the introduction rules for a mutual inductive definition of the different judgment forms, we get a *negative* inductive definition!

Metamathematical account of meaning explanations

- If we interpret the meaning explanation rules as the introduction rules for a mutual inductive definition of the different judgment forms, we get a *negative* inductive definition!
- However, it's an *inductive-recursive* definition. The correct (equal) *types* are *inductively* generated while *simultaneously* defining *recursively* what it means to be (equal) *objects* in them.

Metamathematical account of meaning explanations

- This inductive-recursive definition can be given *set-theoretic meaning*
  - Aczel 1980, *Frege Structures and the Notions of Proposition, Truth and Set*.
  - Allen 1987, *A Non-Type-Theoretic Semantics for Type-Theoretic Language*.

Metamathematical account of meaning explanations

- This inductive-recursive definition can be given *set-theoretic meaning*
  - Aczel 1980, *Frege Structures and the Notions of Proposition, Truth and Set*.
  - Allen 1987, *A Non-Type-Theoretic Semantics for Type-Theoretic Language*.
- Alternatively, a *translation* into Logical Theory of Constructions (intutitionistic predicate logic + untyped lambda calculus + inductive definitions , Aczel 1974, Smith 1978, 1984), also Martin-Löf's "basic logical theory", with meaning explanations in the Siena lectures 1983.

References

1982 *Constructive Mathematics and Computer Programming* - the meaning of the judgments

1984 *Intuitionistic Type Theory* (Padova lecture notes by Sambin, Bibliopolis) - the meaning of the judgments and the justification of the inference rules

1983 *On the meaning of the logical constants and the justifications of the logical laws* (Siena lecture notes) - the meaning of a basic logical theory, emphasizes epistemological aspects

1987 *Philosophical Implications of Type Theory*, Firenze lectures 1987, meaning explanations for the Logical Framework version of Intuitionistic Type Theory, "an idealistic philosophy in the knowledge theoretical sense"

1998 *Truth and knowability* - change, proof is an ontological concept, proof vs demonstration.

Inductive definitions

We have already seen the objects of

$$\emptyset, 1, A+B, \Sigma x : A.B, \Pi x : A.B, N, Wx : A.B, U$$

What is a constructive mathematical object, in general?

The meaning of A type and $a : A$, general pattern

- The meaning of the categorical judgment $\vdash A$ is that A has a canonical type as value. The clauses have the form
  - $A \Rightarrow C(a_1, \ldots, a_n)$ and furthermore ...,

  where C is a type constructor.
- The meaning of the categorical judgment $\vdash a : A$ is that a has a canonical expression of the canonical type denoted by A as value. The clauses have the form
  - $A \Rightarrow C(a_1, \ldots, a_n)$ and $a \Rightarrow c(b_1, \ldots, b_m)$ and furthermore ...,

  where c is a term constructor matching C

and similarly for the equality judgments. What are "furthermore ..."?

Martin-Löf's theory of iterated inductive definitions 1971

Natural number predicate

$$\frac{}{N(0)} \qquad \frac{N(x)}{N(s(x))}$$

Identity relation

$$\frac{}{I(x,x)}$$

The "identical to *a*" predicate

$$\frac{}{I_a(a)}$$

Elimination rules can be derived.

Elimination rules

for the identity relation

$$\frac{\Gamma \vdash_{X,x,x'} I(x,x') \qquad \Gamma \vdash_{X,y} C[x := y, x' := y]}{\Gamma \vdash_{X,x,x'} C}$$

for the "identical to *a*" predicate

$$\frac{\Gamma \vdash_{X,x} I_a(x) \qquad \Gamma \vdash_{X} C[x := a]}{\Gamma \vdash_{X,x} C}$$

A general identity type - introduction rule

Predicate logic

$$\frac{}{I_a(a)}$$

Intuitionistic Type Theory

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash r : I(A, a, a)}$$

A general identity type - elimination rule (Paulin)

Predicate logic

$$\frac{\Gamma \vdash_{X,x} I_a(x) \qquad \Gamma \vdash_X C[x := a]}{\Gamma \vdash_{X,x} C}$$

Intuitionistic Type Theory

$$\frac{\Gamma, x : A \vdash c : I(A, a, x) \qquad \Gamma \vdash d : C[x := a, z := r]}{\Gamma \vdash J(c, d) : C}$$

where

$$\Gamma, x : A, z : I(A, a, x) \vdash C$$

I-equality

$$J(r, d) \quad = \quad d$$

The meaning of the identity type

$\vdash A$ has new clause

- $A \Rightarrow I(B, b, b')$ and $\vdash B$ and $\vdash b, b' : B$.

$\vdash a : A$ has new clause

- $A \Rightarrow I(B, b, b')$ and $a \Rightarrow r$ and $\vdash b = b' : B$.

New rules for identity ("exact equality")

Identity reflection

$$\frac{\Gamma \vdash c : I(A, a, a')}{\Gamma \vdash a = a' : A}$$

Uniqueness of identity proofs:

$$\frac{\Gamma \vdash c : I(A, a, a')}{\Gamma \vdash c = r : I(A, a, a')}$$

These rules destroy the decidability of judgments in Intuitionistic Type Theory.