Algorithmic Analysis of Polygonal Hybrid Systems

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VERIMAG

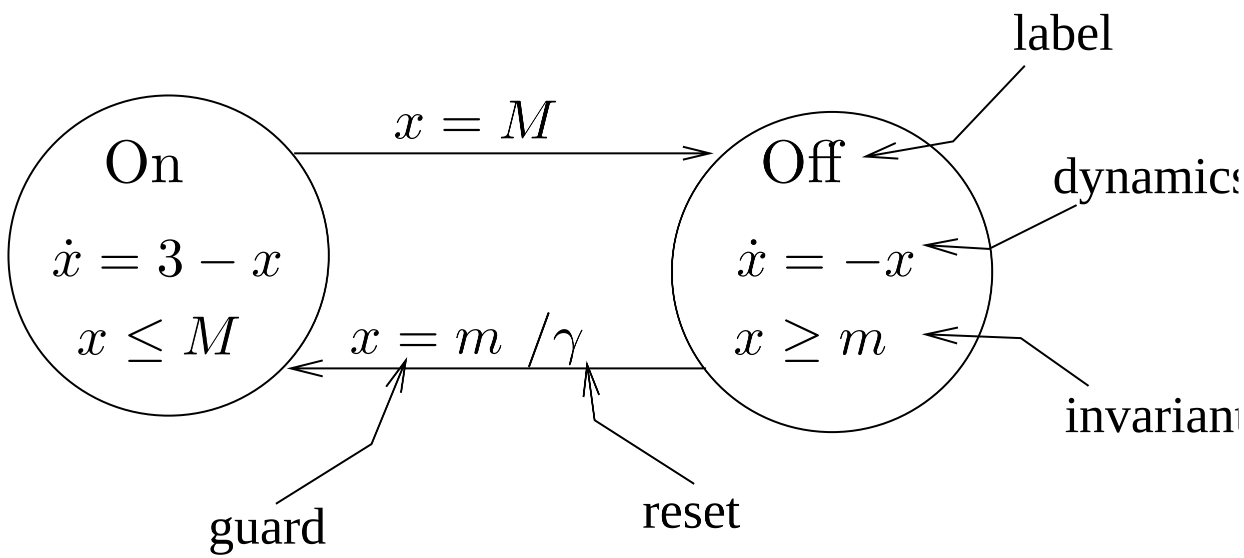
GRENOBLE

Hybrid Systems

- Hybrid Systems: interaction between discrete and continuous behaviors
- Examples: thermostat, automated highway systems, air traffic management systems, robotic systems, chemical plants, etc.

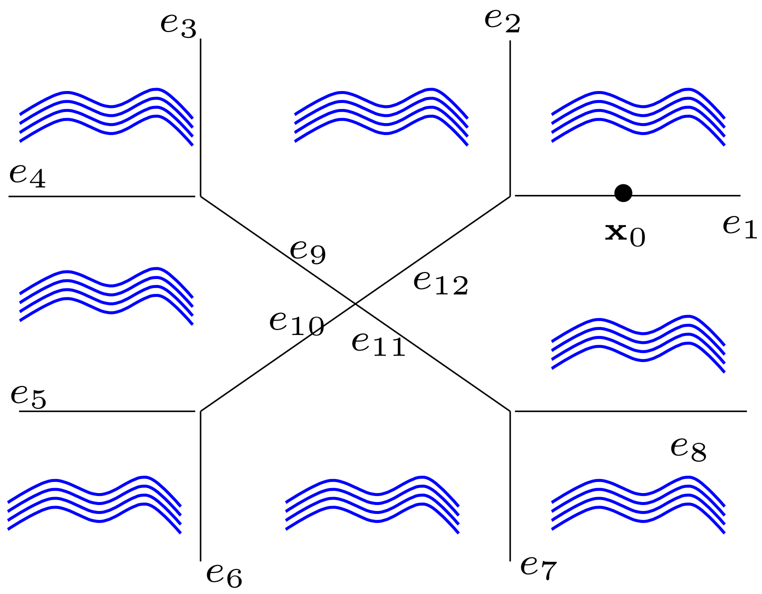
Hybrid Systems

Model: Hybrid Automata



Hybrid Systems

Example: Swimmer in a whirlpool



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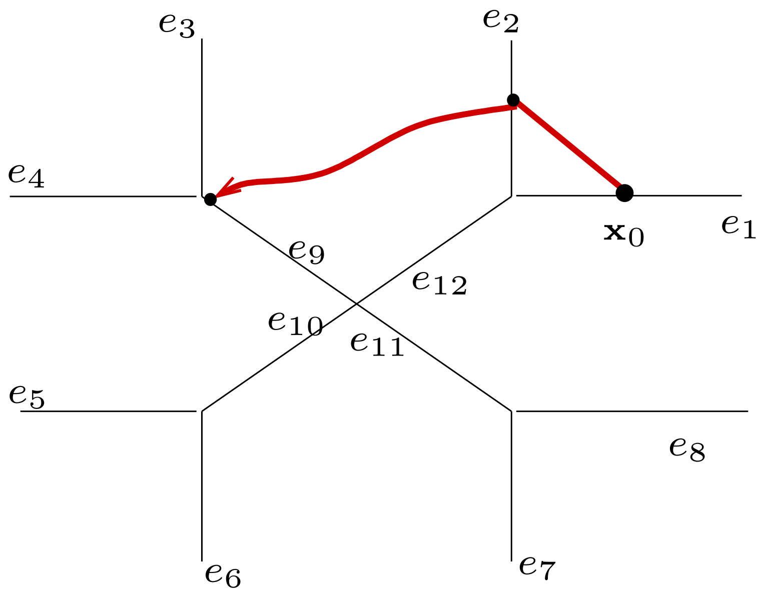

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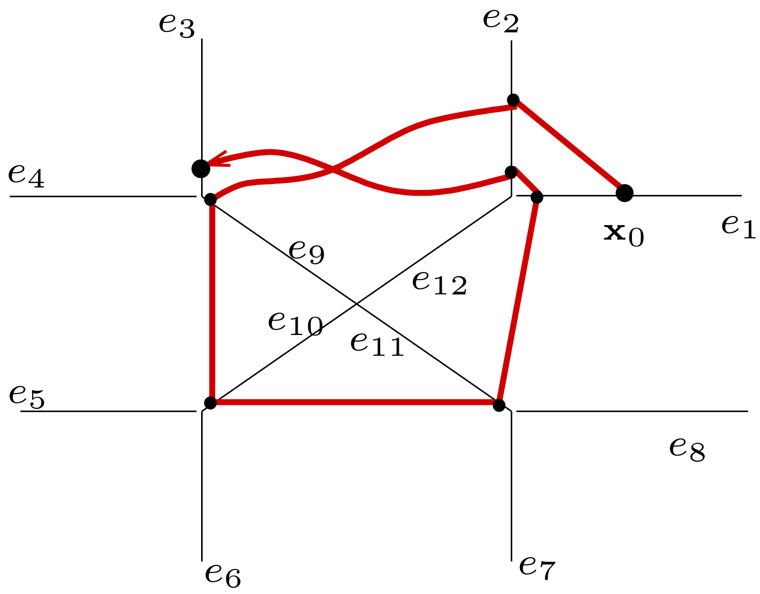

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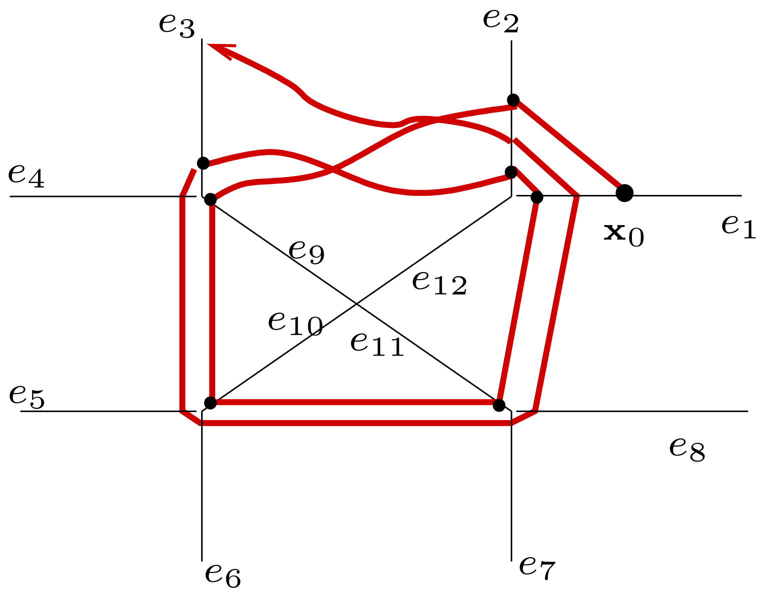

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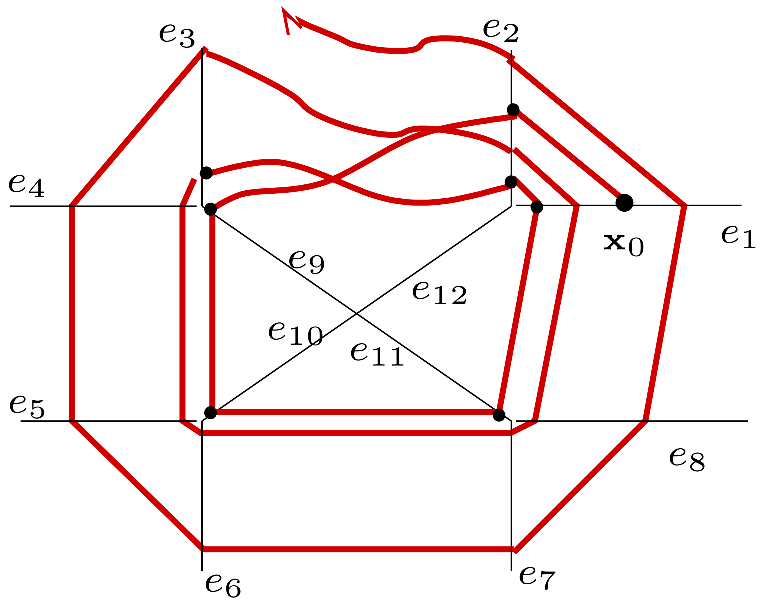

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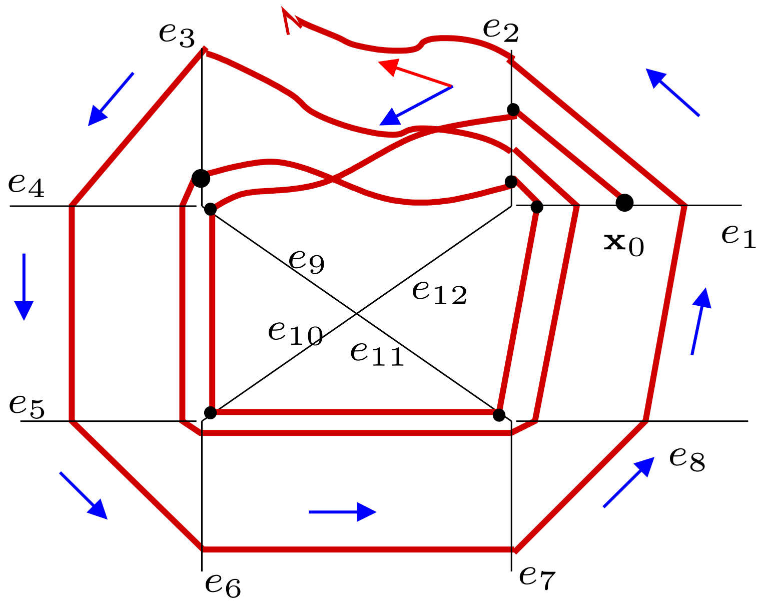
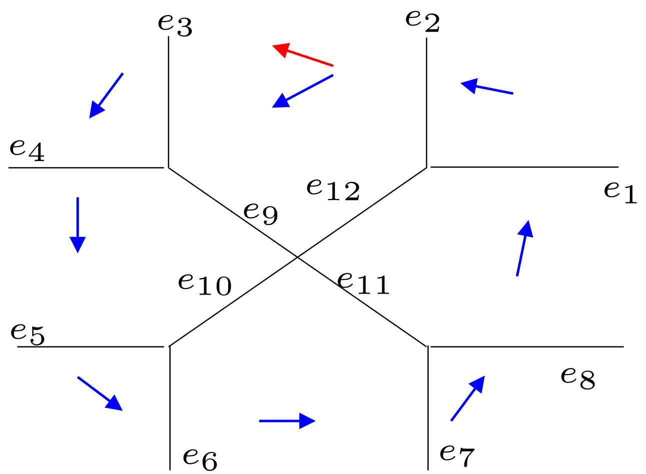


Polygonal Differential Inclusion Systems (SPDIs)

- A partition of the plane into convex polygonal regions
- A constant differential inclusion for each region

$$\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}} \text{ if } \mathbf{x} \in R_i$$



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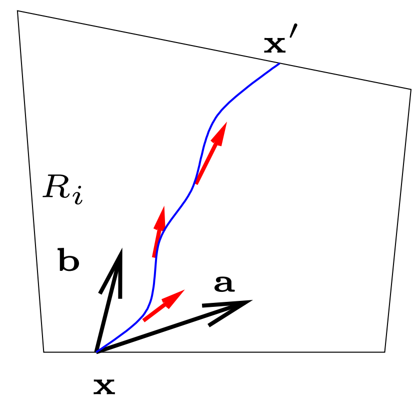

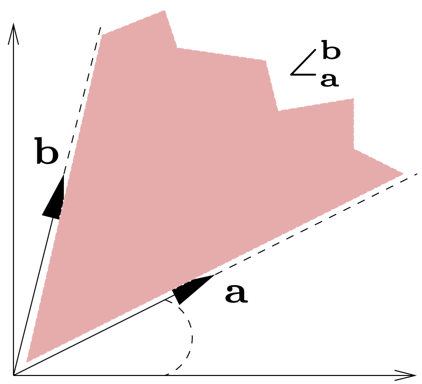

Polygonal Differential Inclusion Systems (SPDIs)

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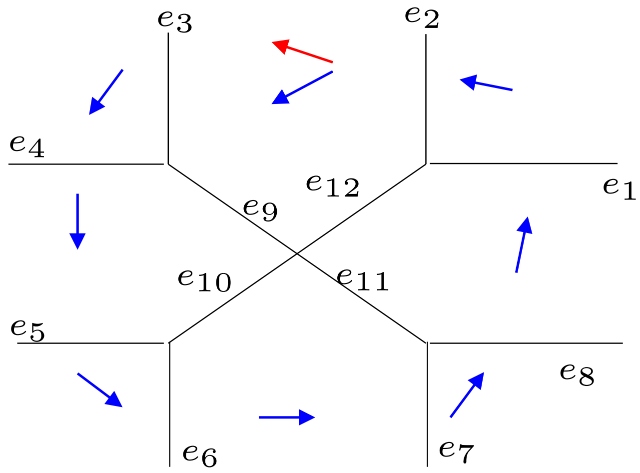

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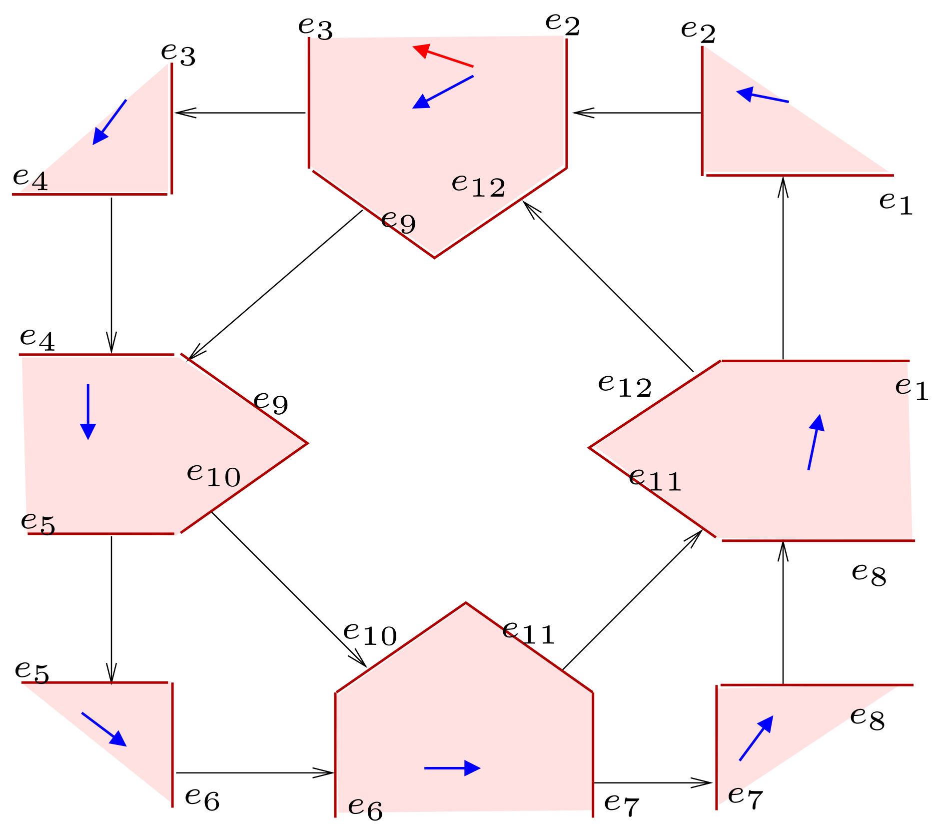

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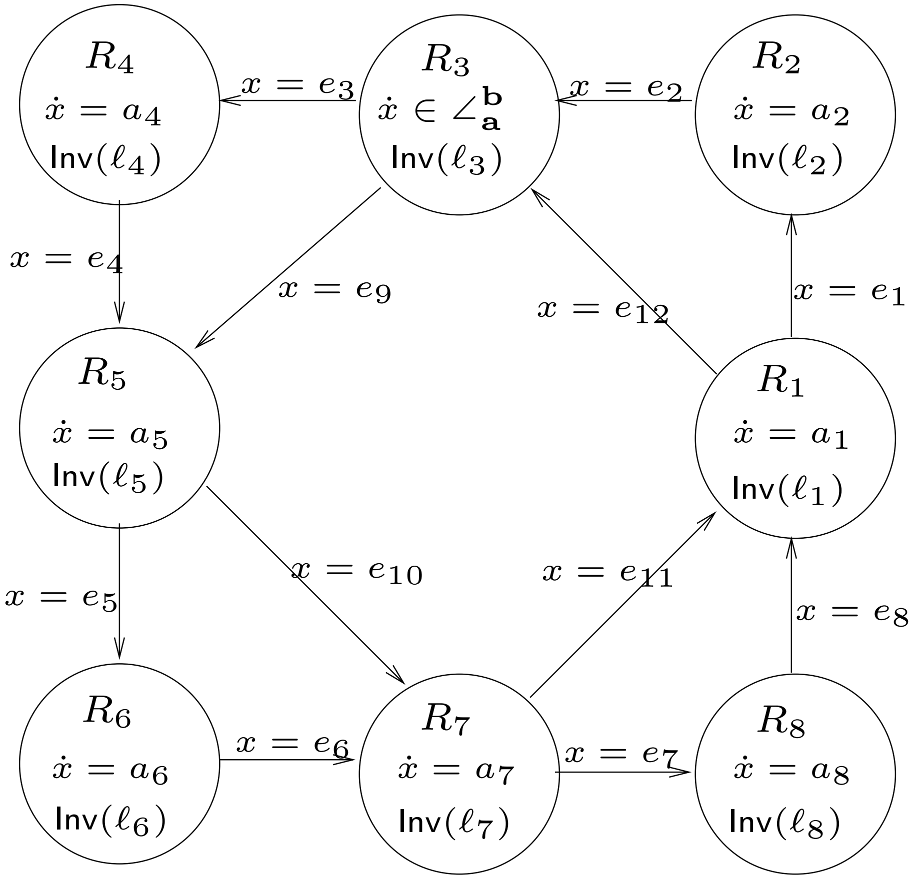


Polygonal Differential Inclusion Systems (SPDIs)

- The “swimmer” is a hybrid system
- Hybrid Automata?

We will use the “geometric” representation instead of the hybrid automata

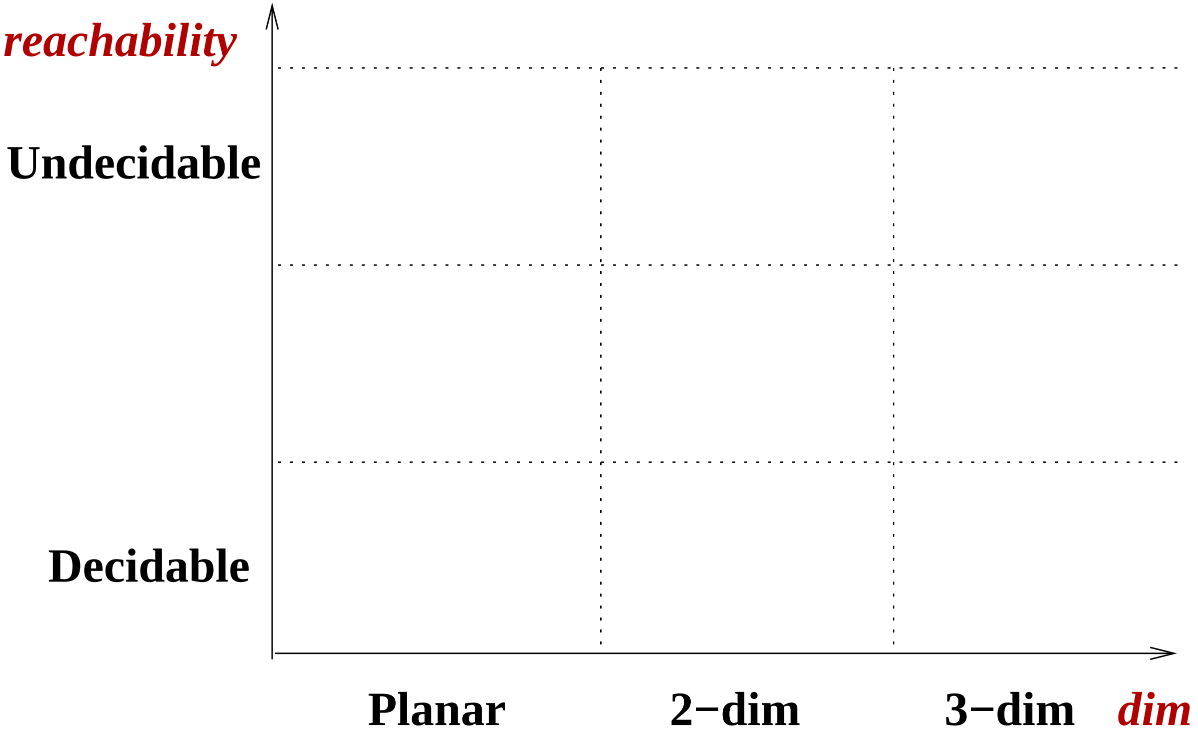
Overview of the presentation

- Motivation and Contributions
- Algorithm for Reachability Problem (SPDIs)
- Implementation – SPeeDI
- Algorithm for Phase Portrait construction (SPDIs)
- Other 2 dim Hybrid Systems
  - Between Decidability and Undecidability
  - Undecidability results
- Summary of Results and Perspectives

Motivation and Contributions

Motivation and Contributions

Scientific Context (Reachability)



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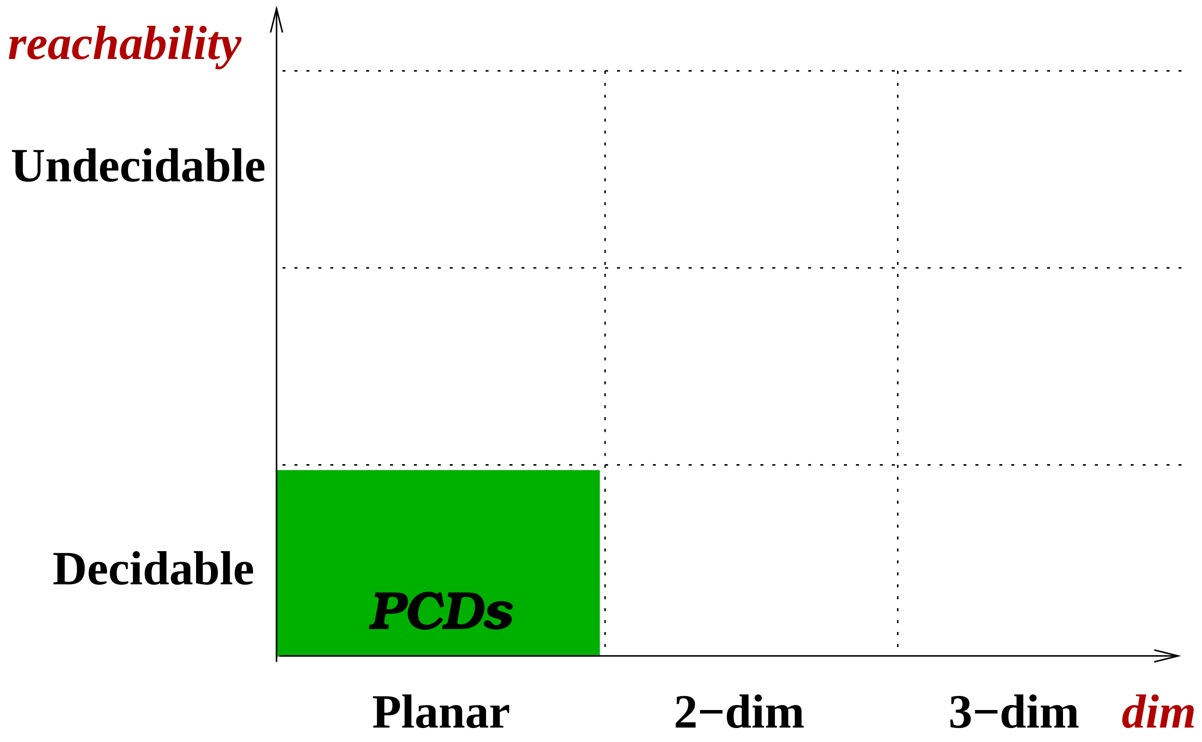

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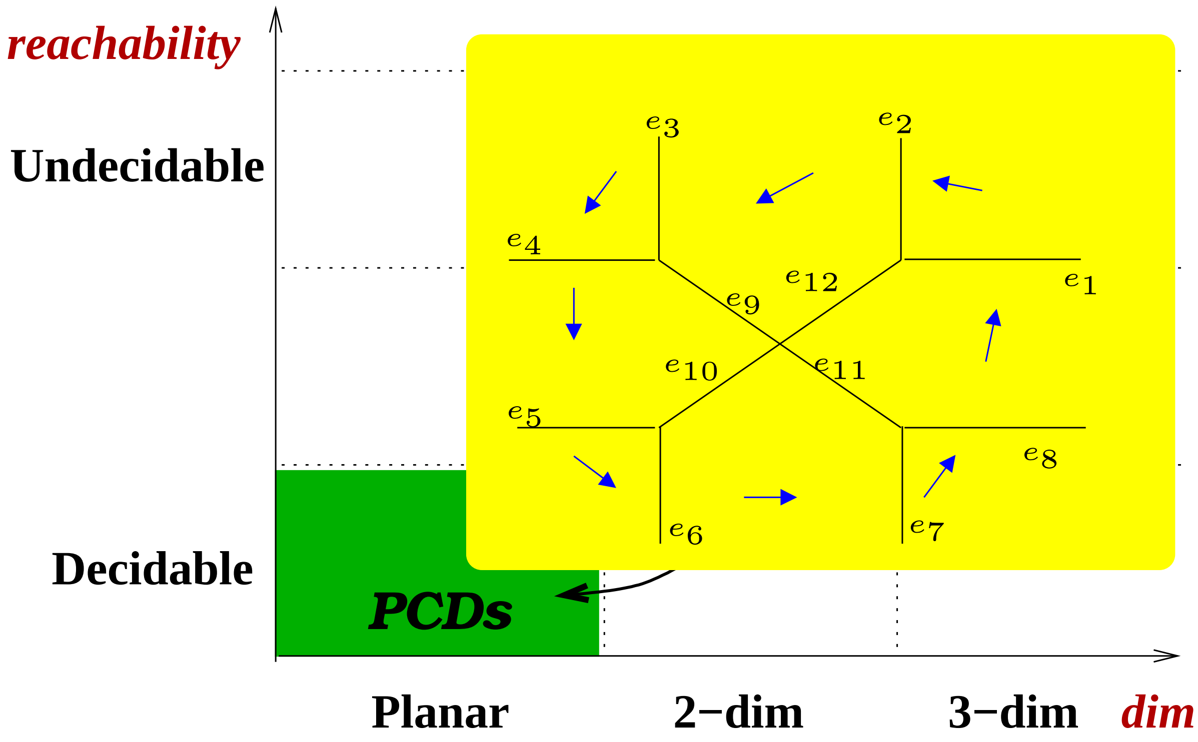

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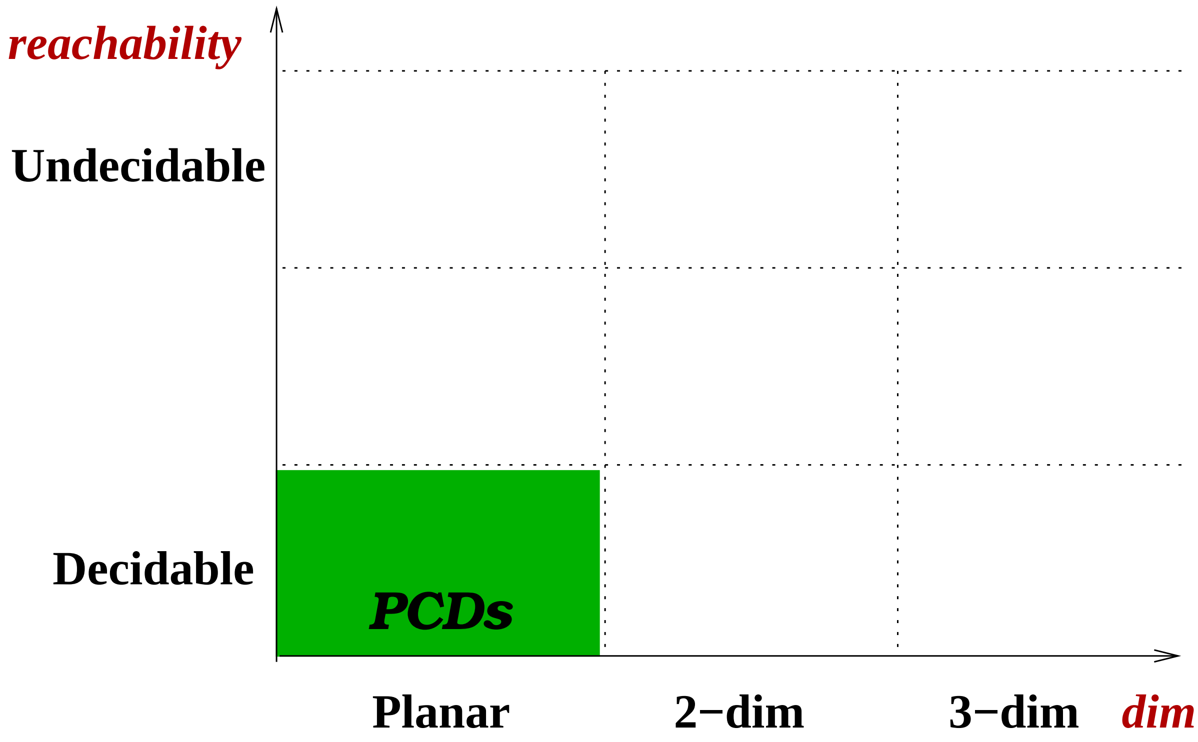

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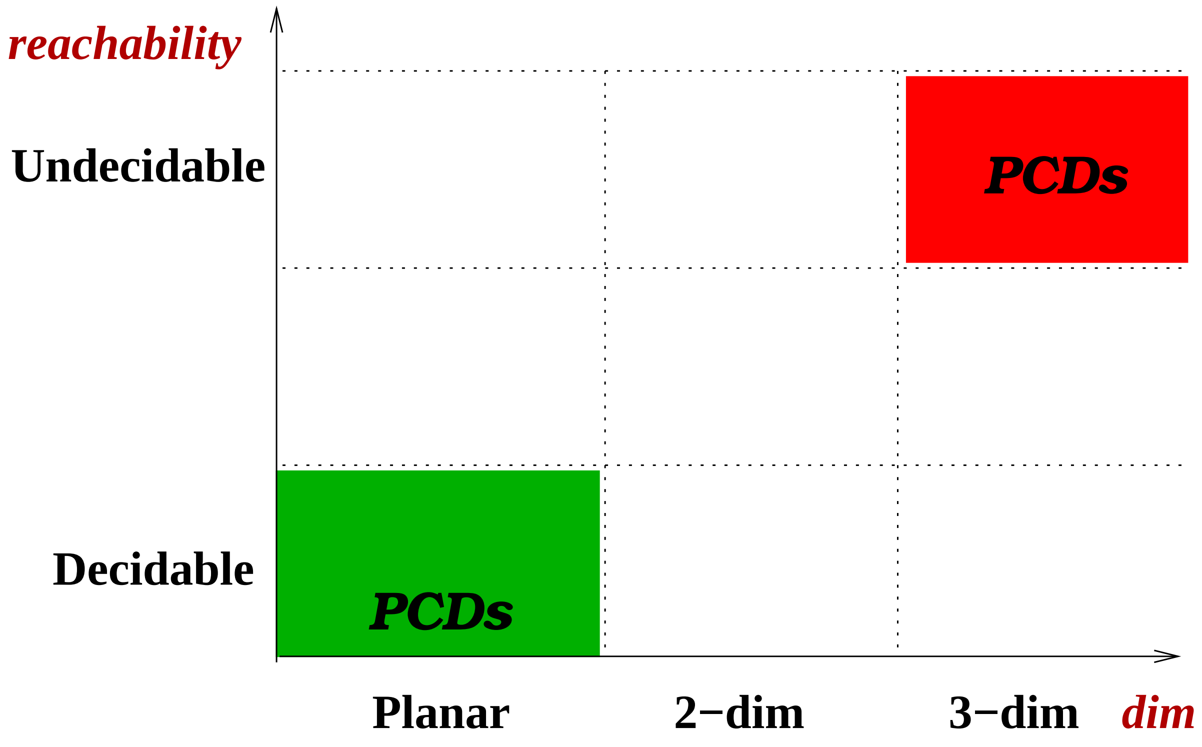

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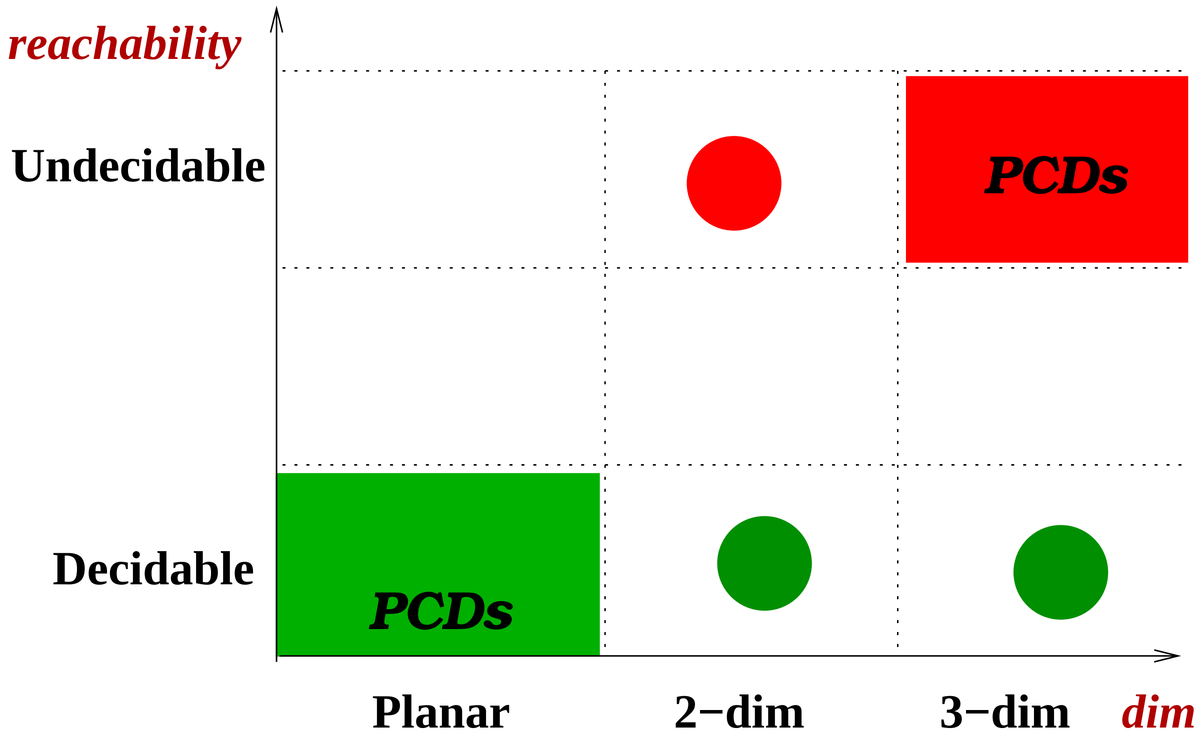
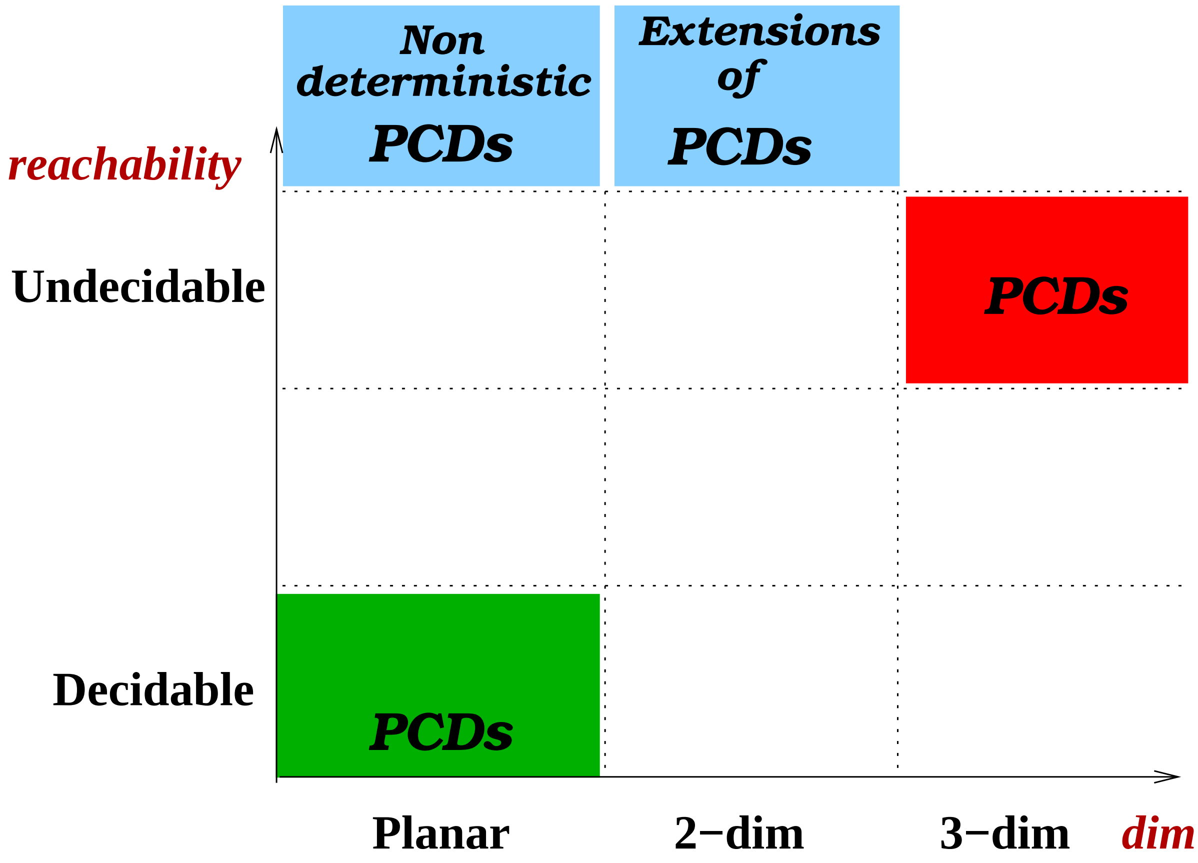


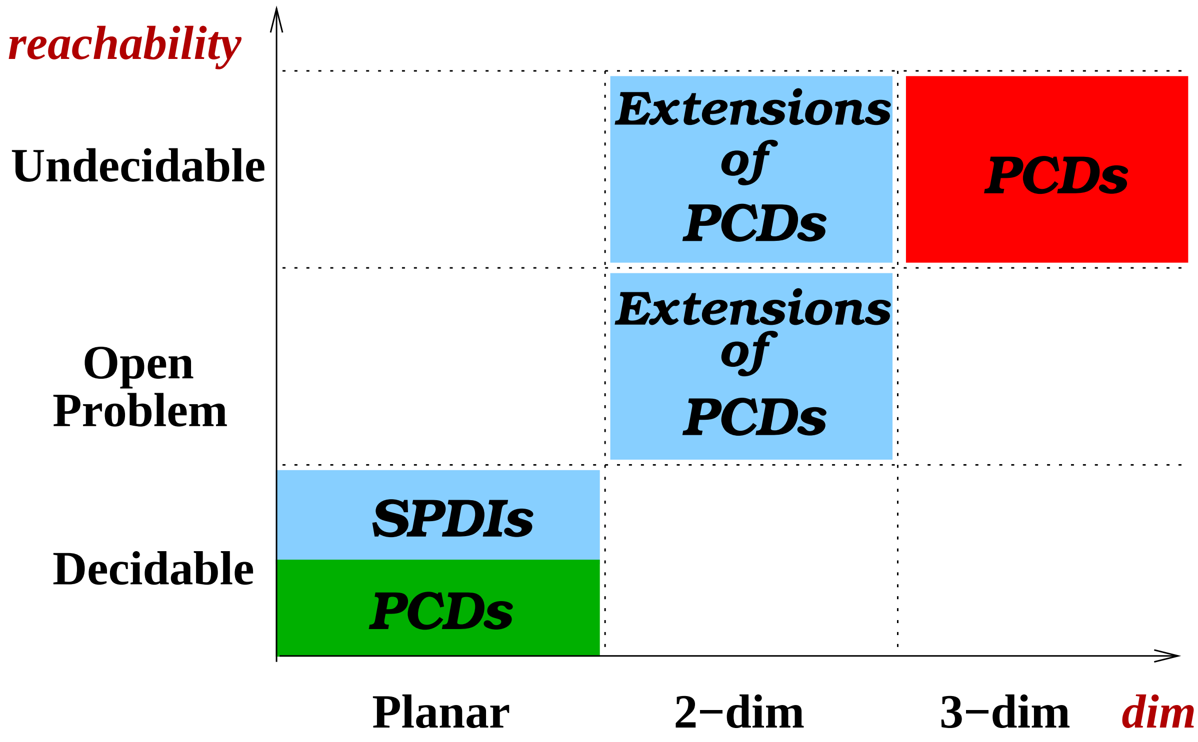
Motivation and Contributions

Challenge



Motivation and Contributions

Contributions (Reachability)



Motivation and Contributions

Scientific Context (Phase Portrait)

- Phase Portrait for PCDs
- Numerical algorithms for computing Viability Kernels

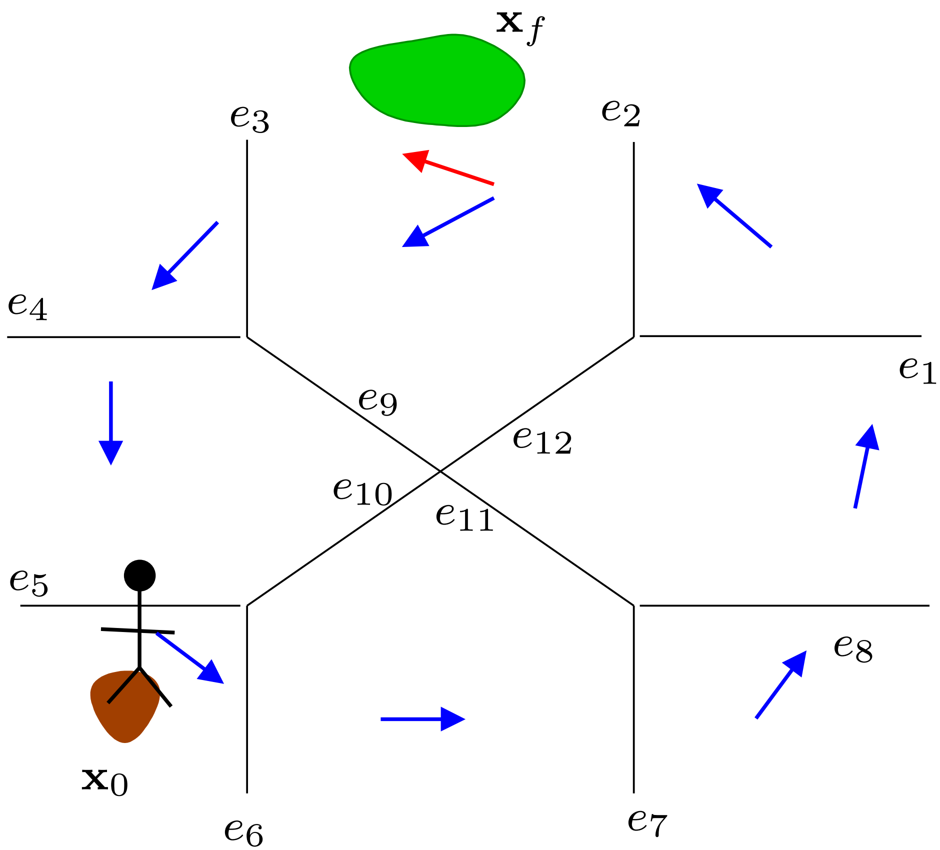
Motivation and Contributions

Contributions (Phase Portrait)

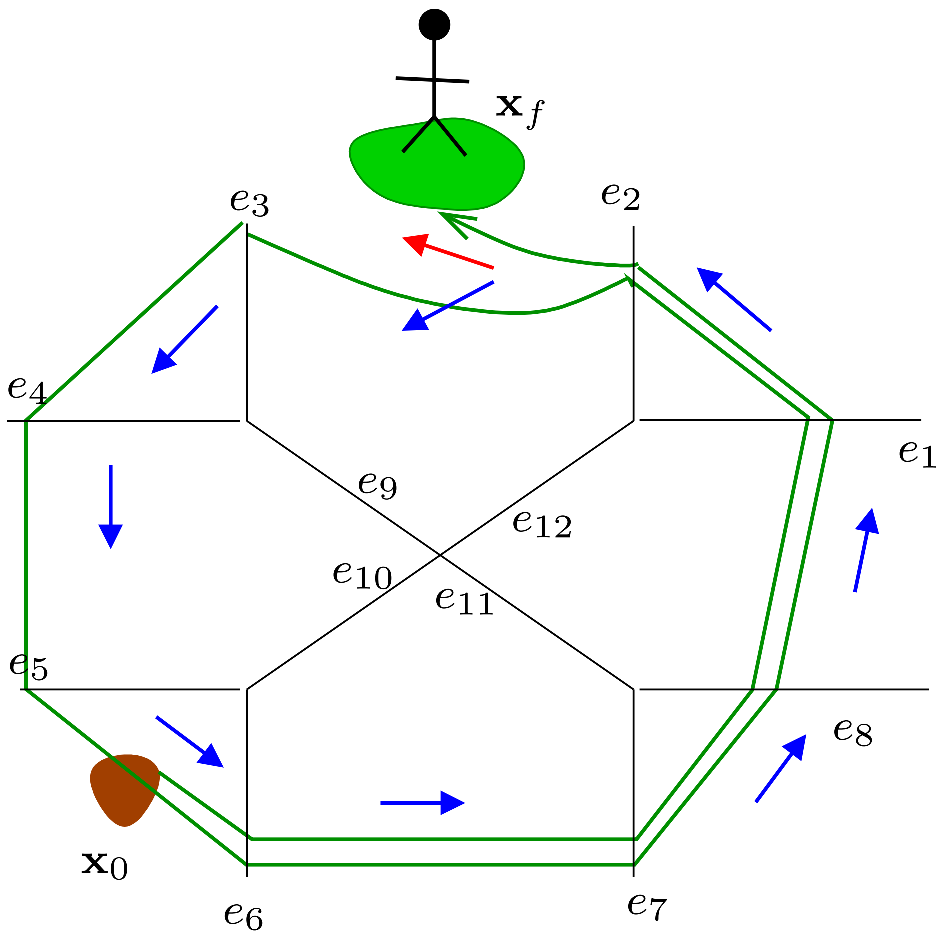
- Phase Portrait for SPDIs
  - Viability Kernel
  - Controllability Kernel

Reachability Analysis for SPDIs

The Reachability Problem for SPDIs



The Reachability Problem for SPDIs

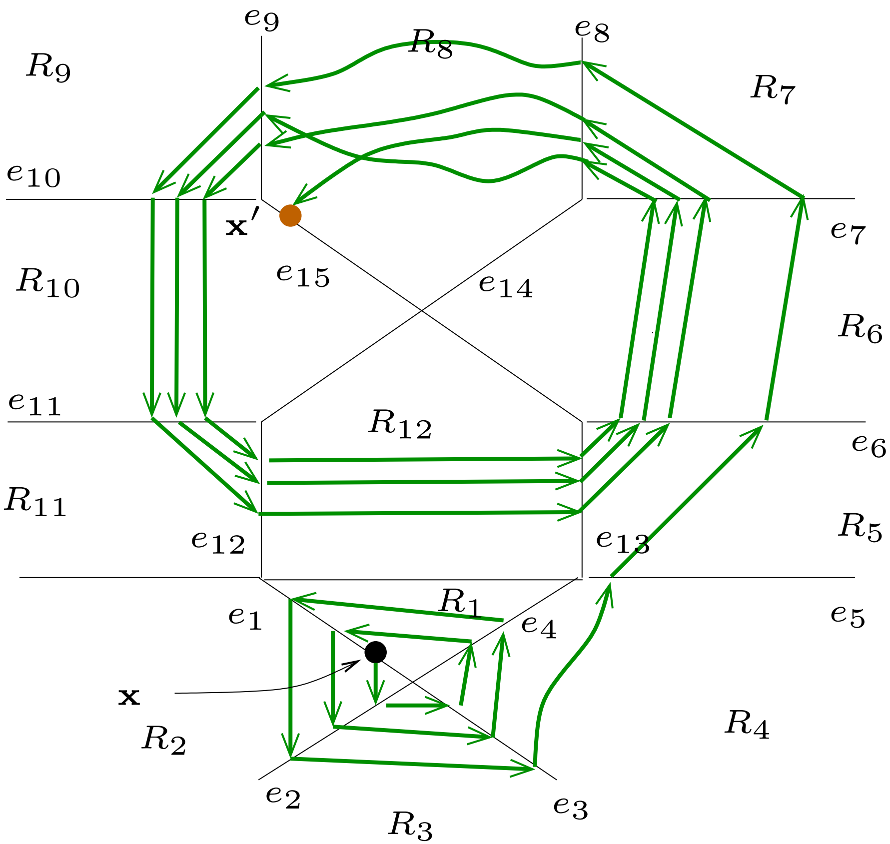


Reachability problem: Is there a trajectory from $\mathbf{x}_0$ to $\mathbf{x}_f$?

Solving the Reachability Problem

1. From trajectories to *simplified trajectories*

1. Simplification of trajectories



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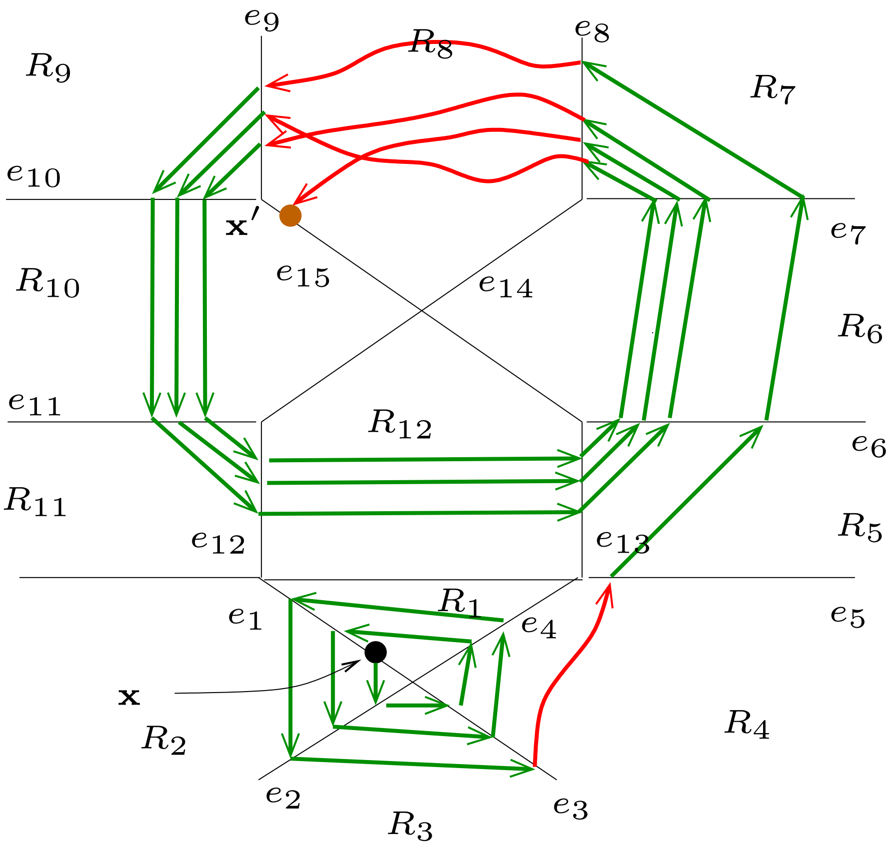

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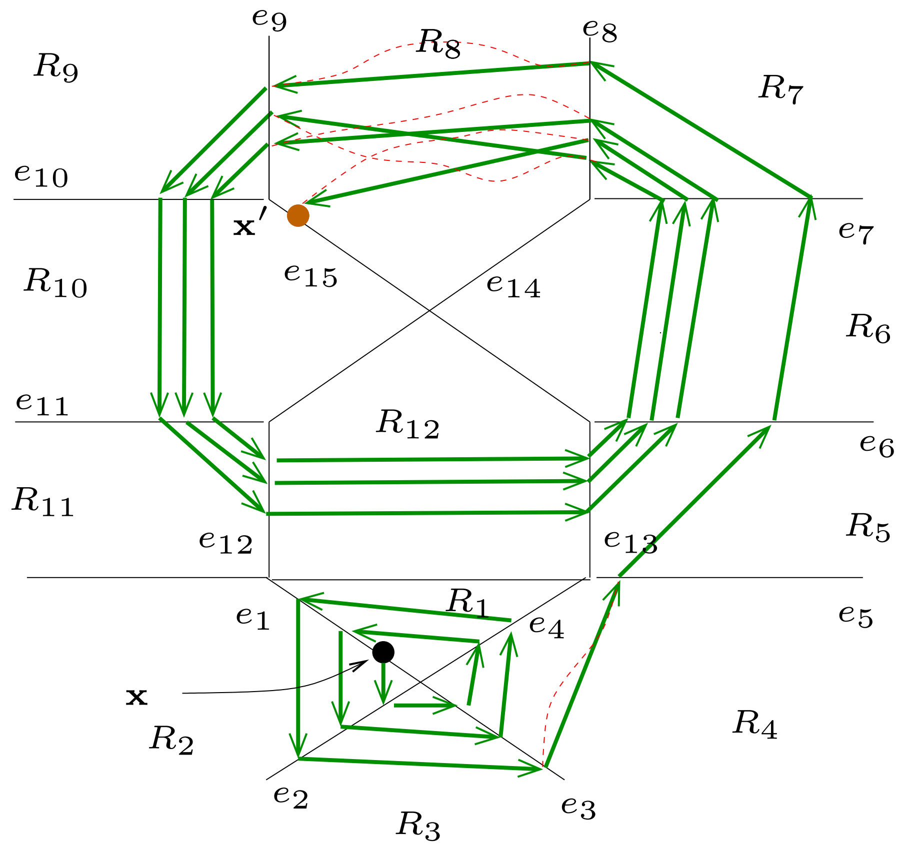

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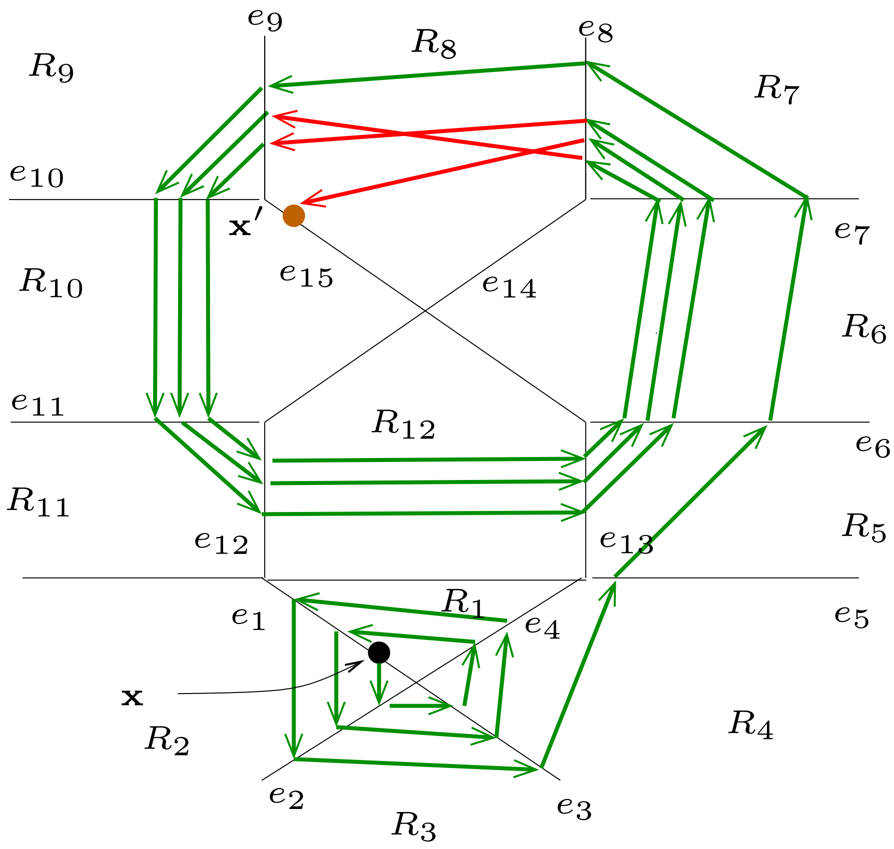

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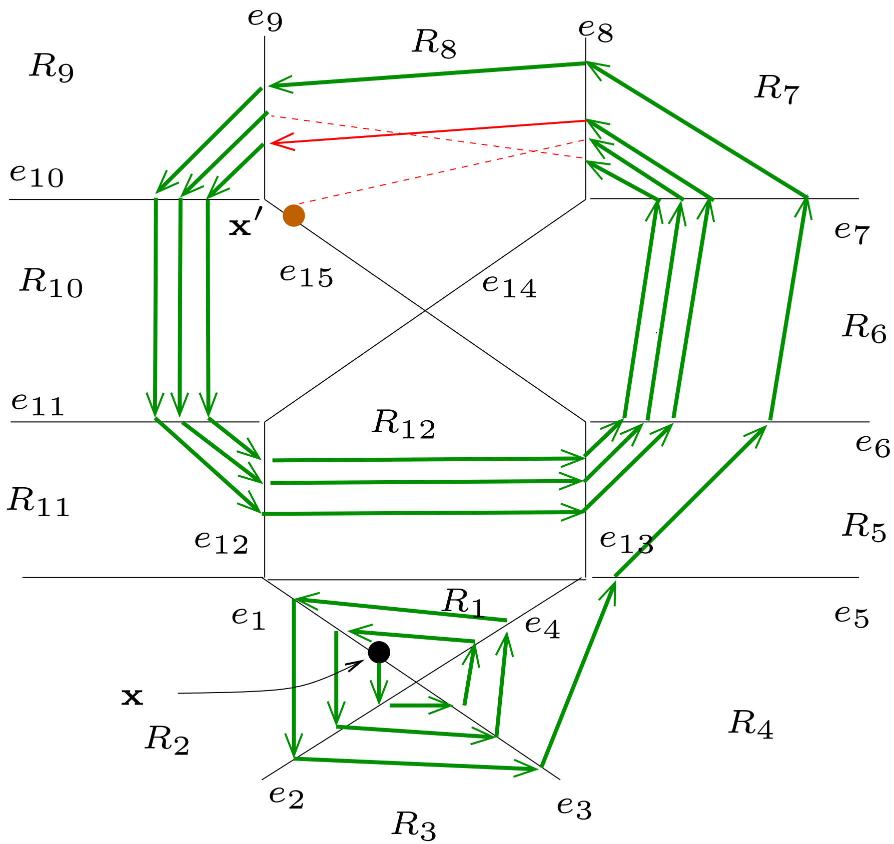

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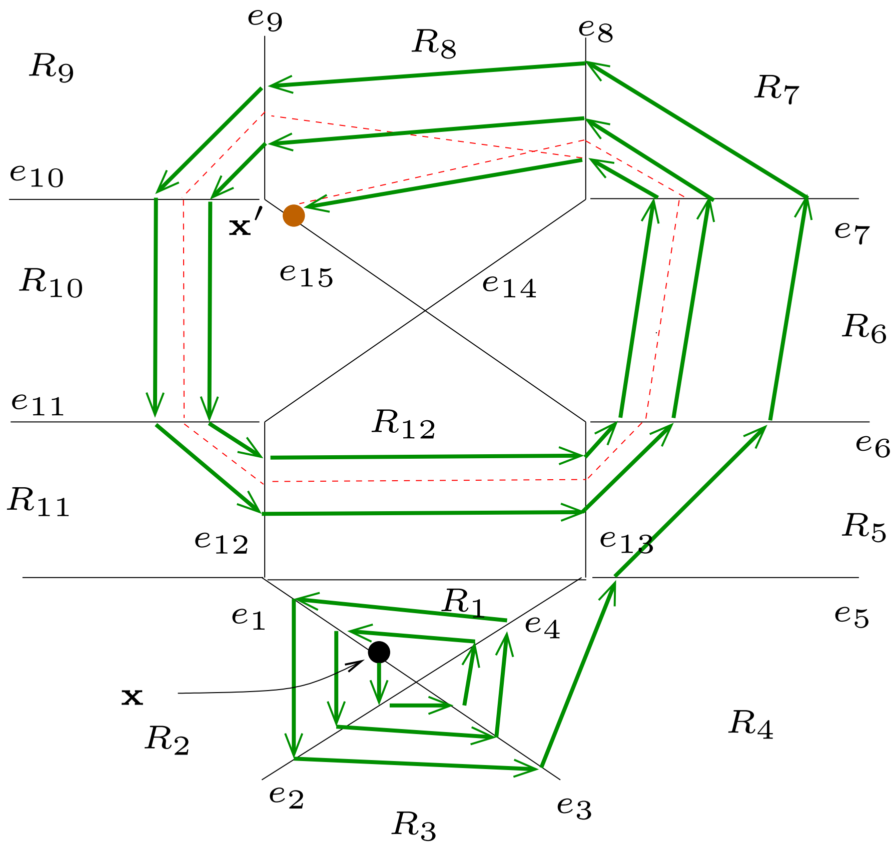

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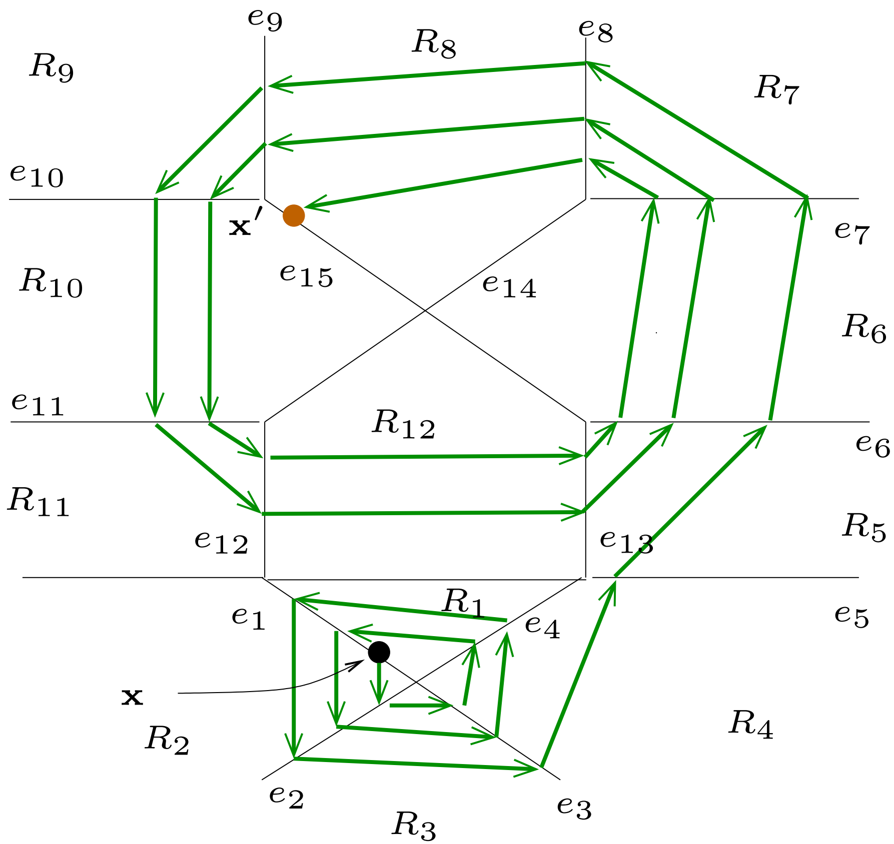
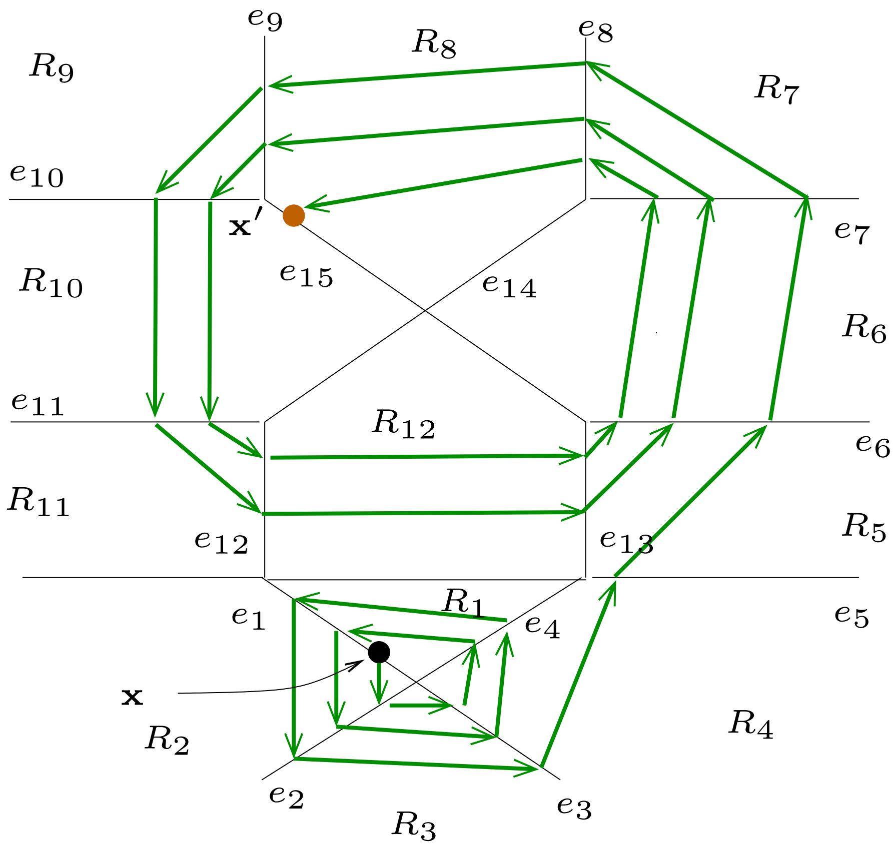


Theorem: If there is an arbitrary trajectory between two points then it always exists a straightened non–crossing trajectory between them

Solving the Reachability Problem

1. From trajectories to *simplified trajectories*
2. From simplified trajectories to *signatures*

2. Abstraction into *signatures*



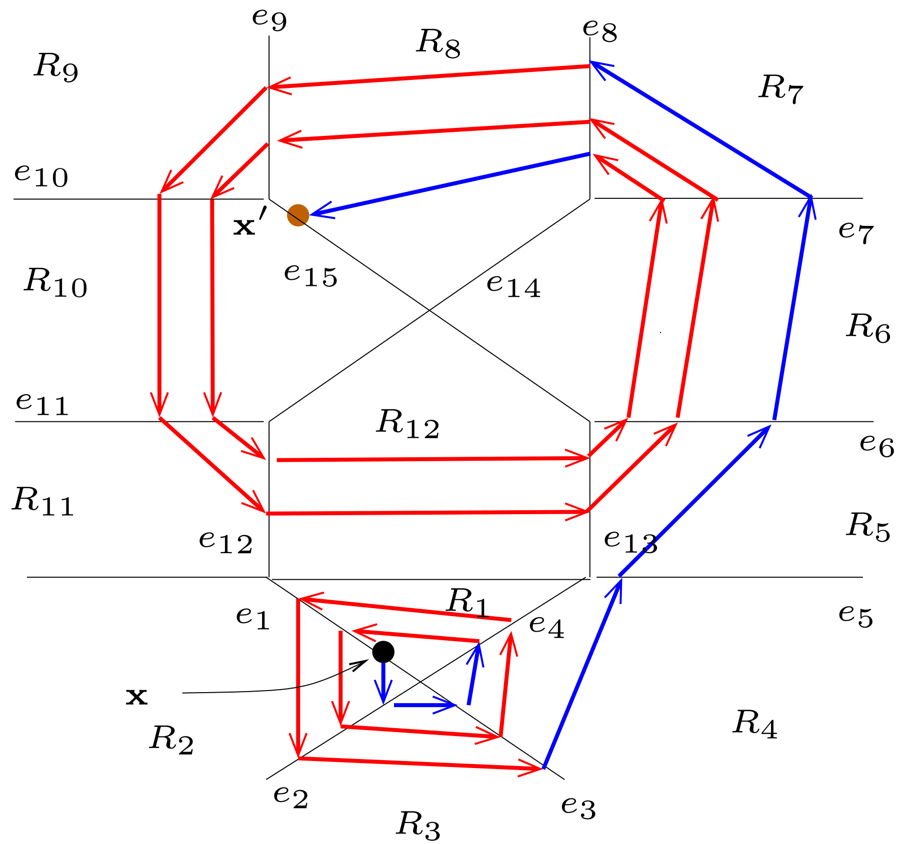
$$\sigma = e_1 e_2 e_3 \ldots e_5 e_6 e_7 \ldots e_{13} e_6 e_7 e_8 e_{15}$$

Solving the Reachability Problem

1. From trajectories to *simplified trajectories*
2. From simplified trajectories to *signatures*
3. From signatures to *factorized signatures*

3. Factorization of Signatures

For $\sigma = e_1 e_2 e_3 \ldots e_5 e_6 e_7 \ldots e_{13} e_6 e_7 e_8 e_{15}$



We obtain the representation:

$\sigma = e_1 e_2 e_3 \left(e_4 e_1 e_2 e_3\right)^2 e_5 e_6 e_7 e_8 \left(e_9 \cdots e_{13} e_6 e_7 e_8\right)^2 e_{15}$

Verimag

3. Canonical Factorization of Signatures

Representation Theorem: Any edge signature $\sigma = e_1, e_2, \ldots, e_n$ can be represented as

$$\sigma = r_1(s_1)^{k_1} r_2(s_2)^{k_2} \ldots r_n(s_n)^{k_n} r_{n+1}$$

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- Properties:
  - r_i is a seq. of pairwise different edges;
  - s_i is a simple cycle;
  - r_i and r_j are disjoint
  - s_i and s_j are different

Proof based on topological properties of the plane

Solving the Reachability Problem

1. From trajectories to *simplified trajectories*
2. From simplified trajectories to *signatures*
3. From signatures to *factorized signatures*
4. From factorized signatures to *types of signatures*

4. Types of Signatures

Abstraction: Any edge signature

$$\sigma = r_1(s_1)^{k_1} r_2(s_2)^{k_2} \ldots r_n(s_n)^{k_n} r_{n+1}$$

belongs to a type

$$type(\sigma) = r_1, s_1, r_2, s_2, \ldots r_n, s_n, r_{n+1}$$

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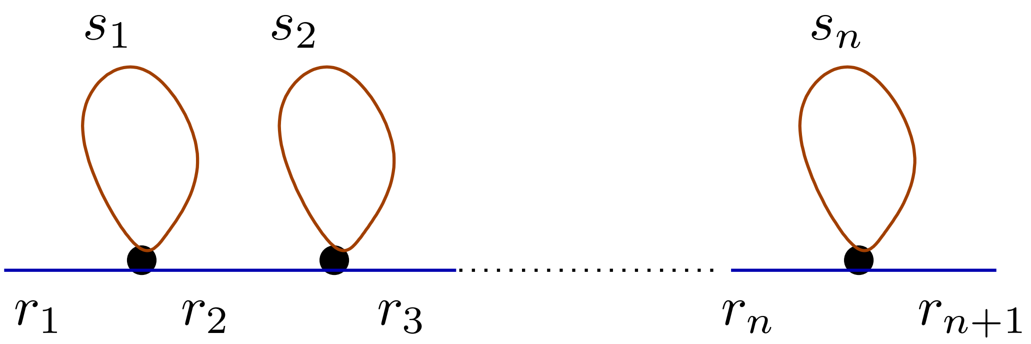

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In the previous example:

$type(\sigma) =$

$$e_1 e_2 e_3,\ e_4 e_1 e_2 e_3,\ e_5 e_6 e_7 e_8,\ e_9 \cdots e_{13} e_6 e_7 e_8,\ e_{15}$$

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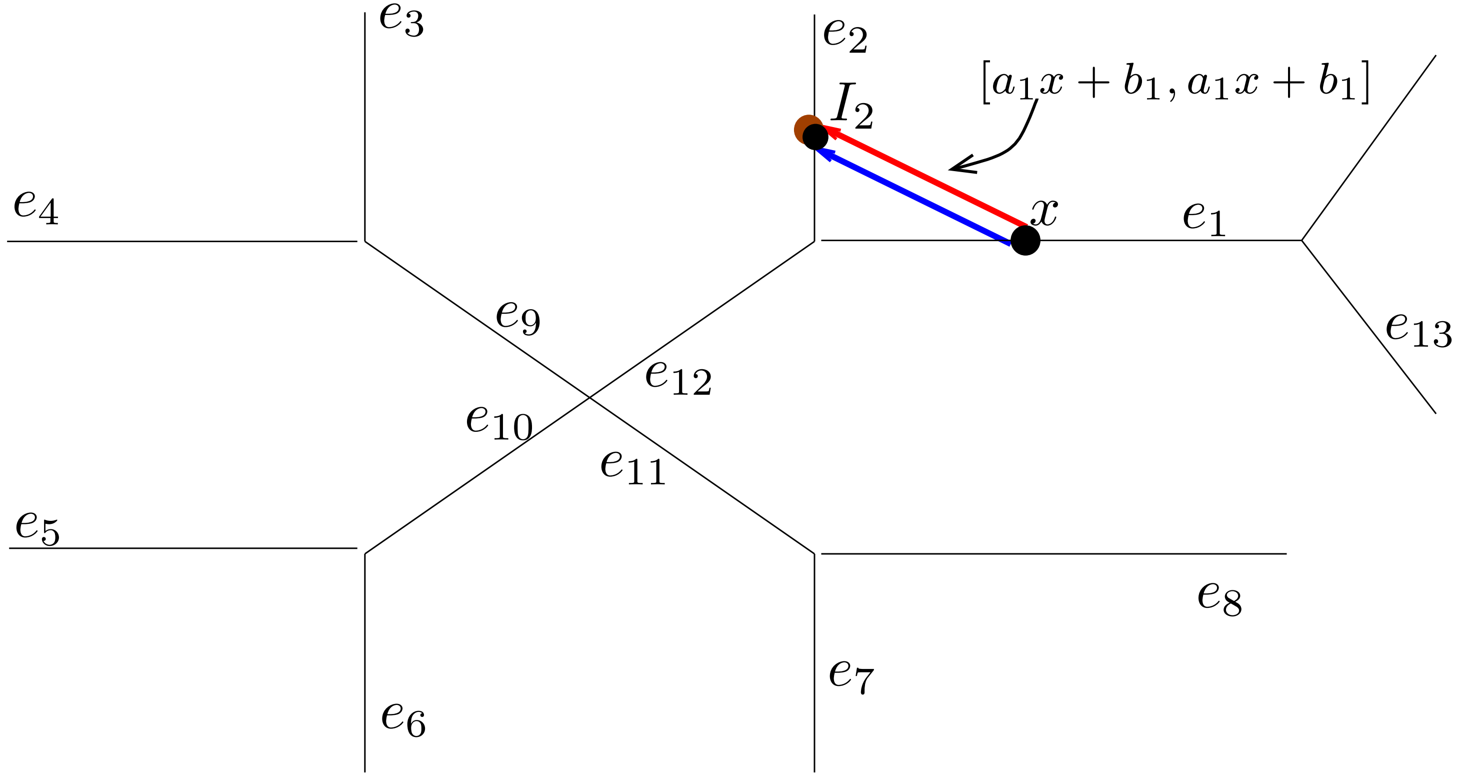
Prop. The set of types of signatures is finite

Solving the Reachability Problem

1. From trajectories to *simplified trajectories*
2. From simplified trajectories to *signatures*
3. From signatures to *factorized signatures*
4. From factorized signatures to *types of signatures*
5. Analysis of each type of signature (computing successors)

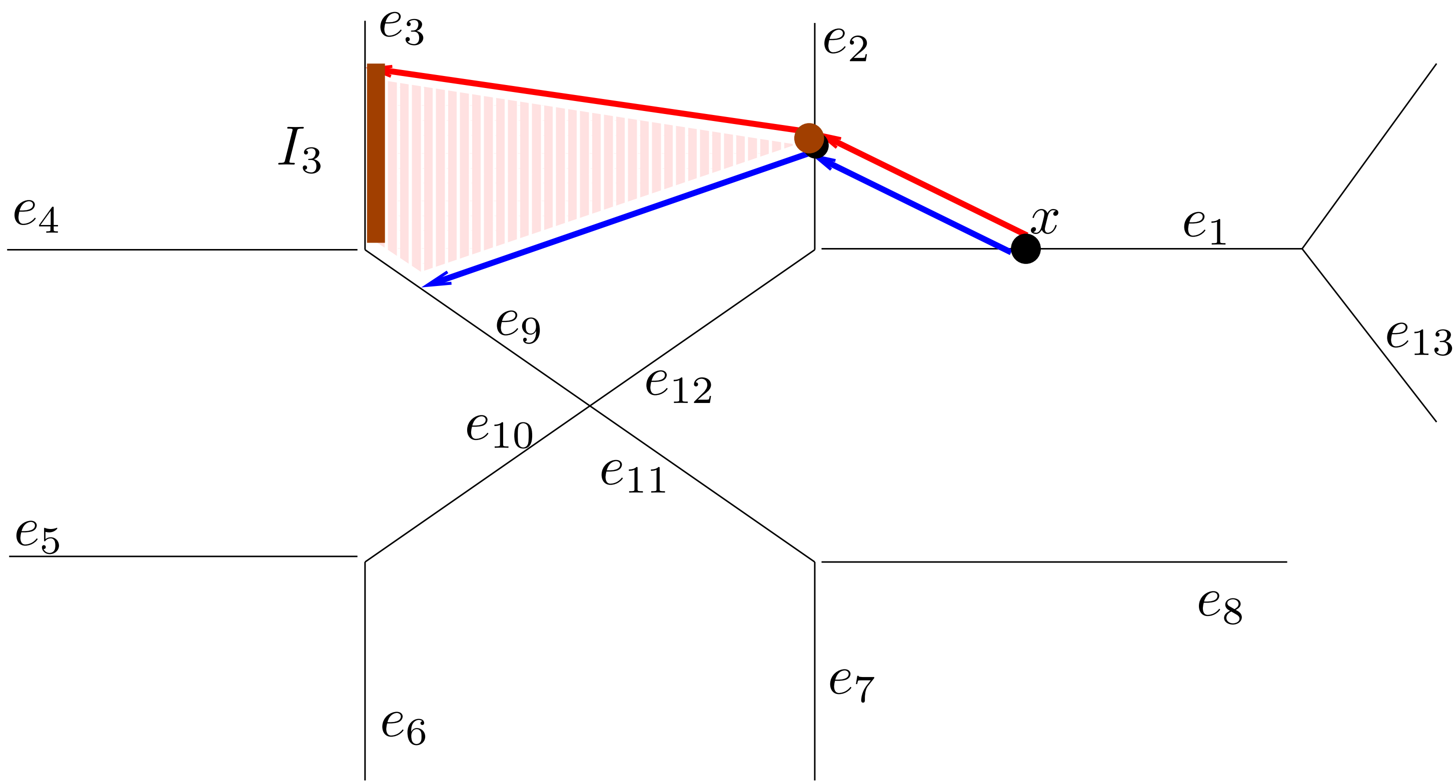
Computing Successors (for σ)

One step ($\sigma = e_1 e_2$)



Computing Successors (for σ)

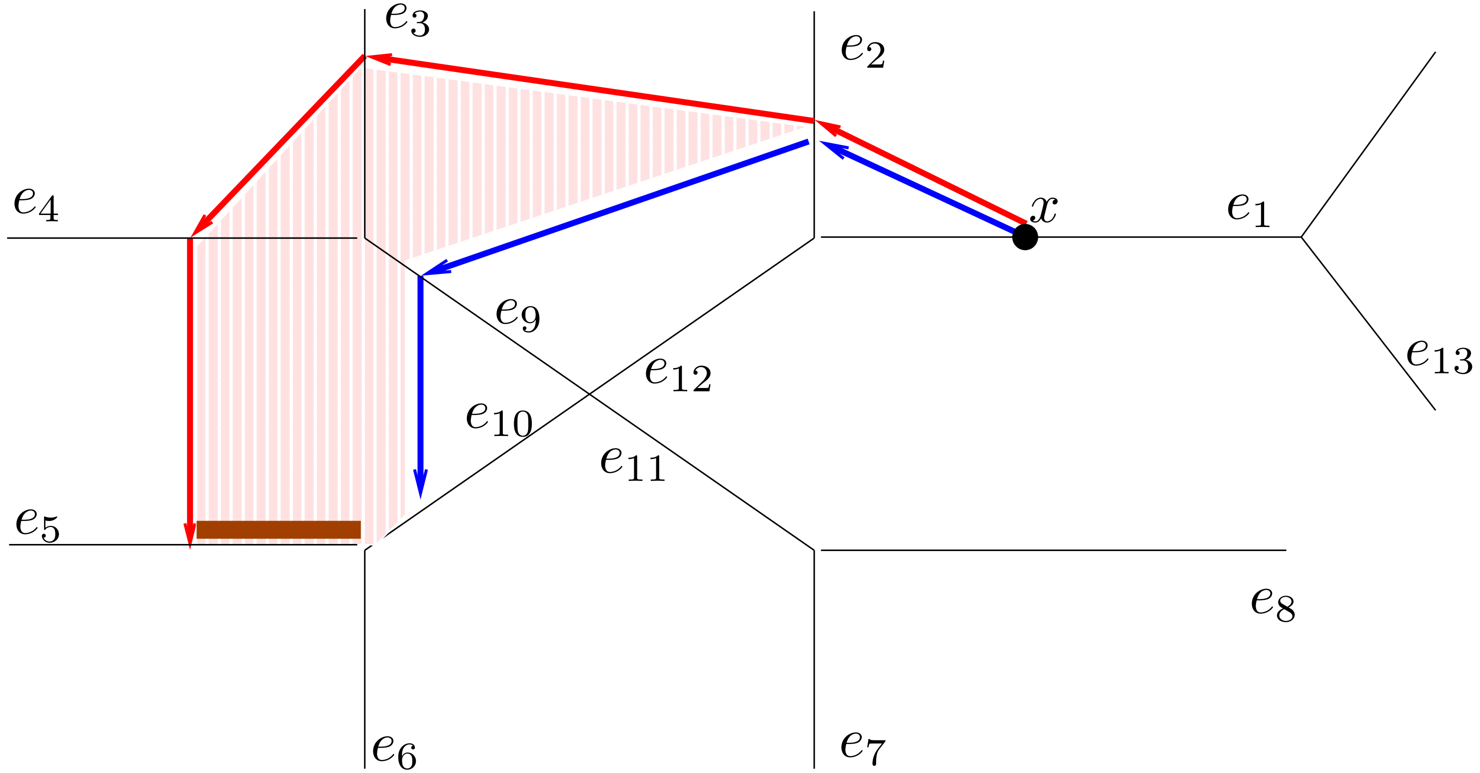
Several steps ($\sigma = e_1 e_2 e_3$)



$$I_3 = \mathsf{Succ}_\sigma(x) = [a_2 x + b_2, a_2' x + b_2'] \cap e_3$$

Computing Successors (for σ)

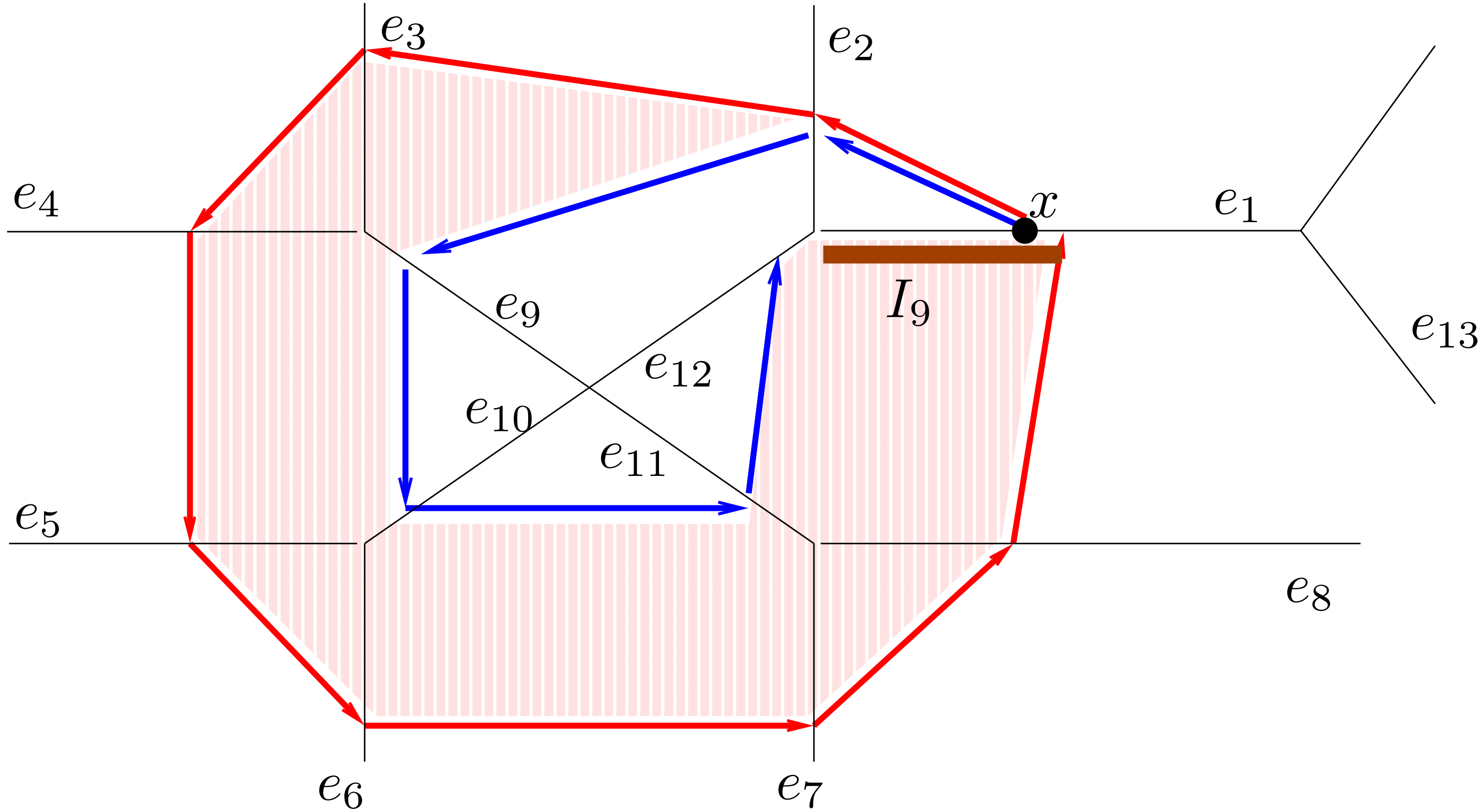
Several steps ($\sigma = e_1 e_2 e_3 e_4 e_5$)



$$I_5 = \mathsf{Succ}_\sigma(x) = [a_4 x + b_4, a'_4 x + b'_4] \cap e_5$$

Computing Successors (for σ)

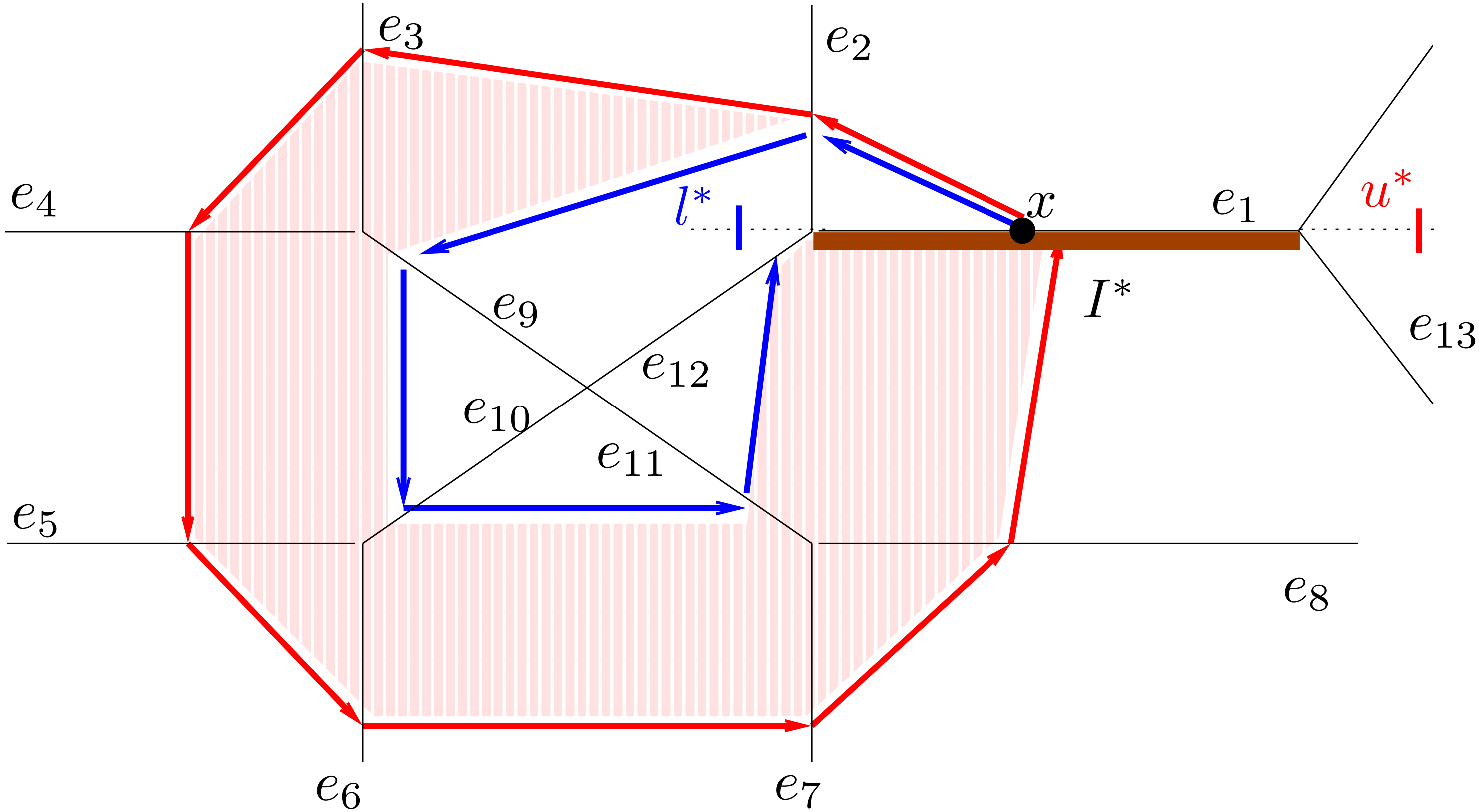
One cycle ($\sigma = s = e_1 e_2 \cdots e_8 e_1$)



$$I_9 = \mathsf{Succ}_\sigma(x) = [a_8 x + b_8, a_8' x + b_8'] \cap e_1$$

Computing Successors (for σ)

One cycle ($\sigma = s = e_1 e_2 \cdots e_8 e_1$)

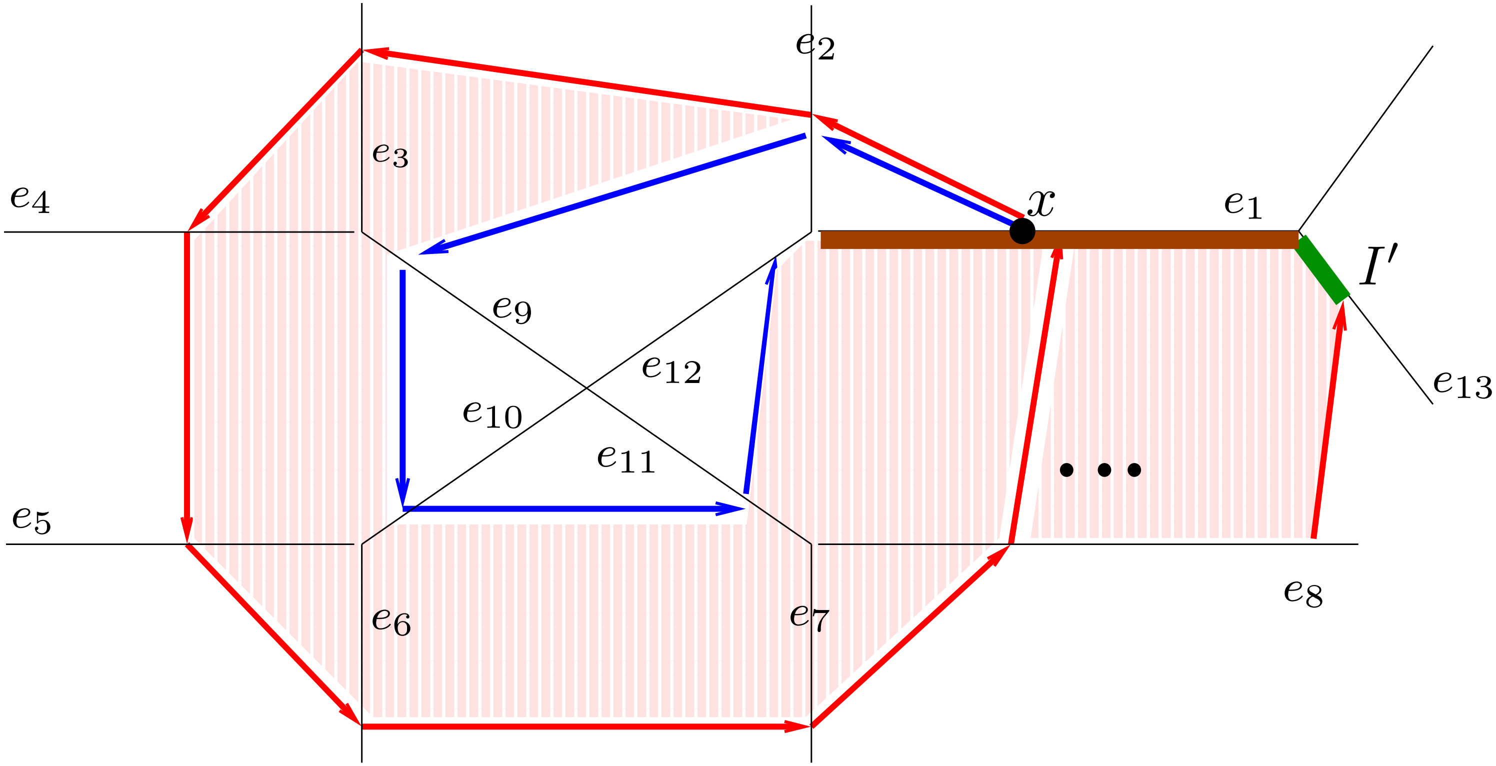


$$l^* = a_1 l^* + b_1 \qquad u^* = a_2 u^* + b_2$$

$$I^* = \mathsf{Succ}^*_\sigma(x) = [l^*, u^*] \cap e_1$$

Computing Successors (for σ)

$\sigma = (s)^* e_{13} \;\; (s = e_1 e_2 \cdots e_8 e_1)$



One cycle iterated: *solution of fixpoint equation (acceleration)*: $I' = \mathsf{Succ}_{e_8 e_{13}} \circ \mathsf{Succ}_{e_1 \cdots e_8} \circ \mathsf{Succ}_s^*(x)$

Computing Successors

Lemma: Successors have the form

$$\mathsf{Succ}_\sigma(l, u) = [a_1 l + b_1, a_2 u + b_2] \cap J \text{ if } [l, u] \subseteq S$$

Lemma: Fixpoint equations

$$[a_1 l^* + b_1, a_2 u^* + b_2] = [l^*, u^*]$$

can be explicitely solved (without iterating).
We have that ($I = [l, u]$):

$$\mathsf{Succ}^*_\sigma(I) = [l^*, u^*] \cap J$$

Reachability Algorithm

for each type of signature τ **do**
check whether $Reach_\tau(x_0, x_f)$

To test whether $Reach_\tau(x_0, x_f)$ for

$$\tau = r_1(s_1)^* \cdots (s_n)^* r_{n+1}$$

Compute Succ_r
Accelerate $(\mathsf{Succ}_s)^*$

Reachability: Main Result

- The capability of computing fixpoints for simple cycles (acceleration)
- The set of types of signatures is finite

Reachability is decidable for SPDI

SPeeDI: a Tool for SPDIs

Implementation: SPeeDI

- We have implemented the reachability algorithm for SPDIs: SPeeDI (joint work with Gordon Pace)
- Language: Haskell

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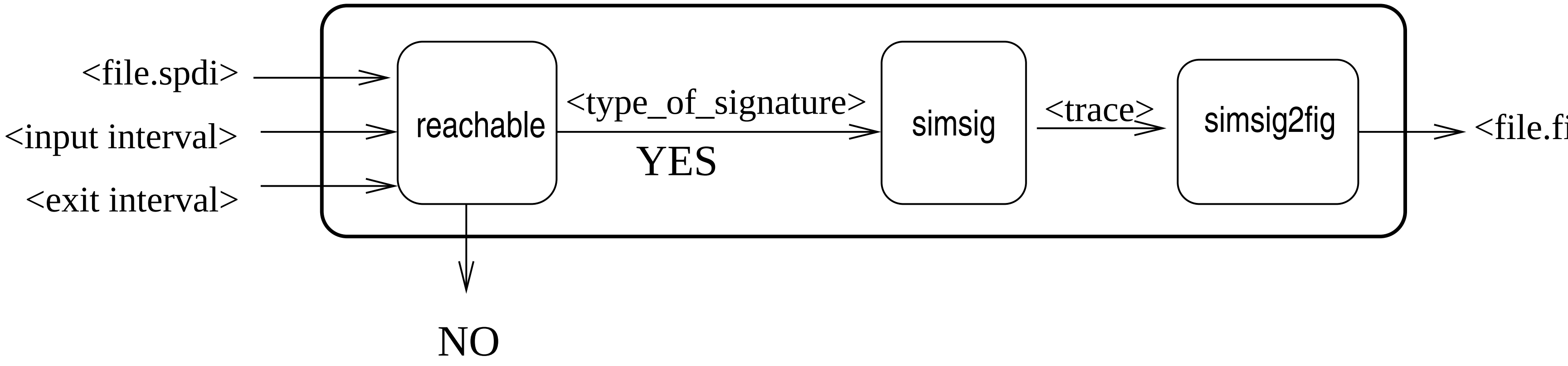

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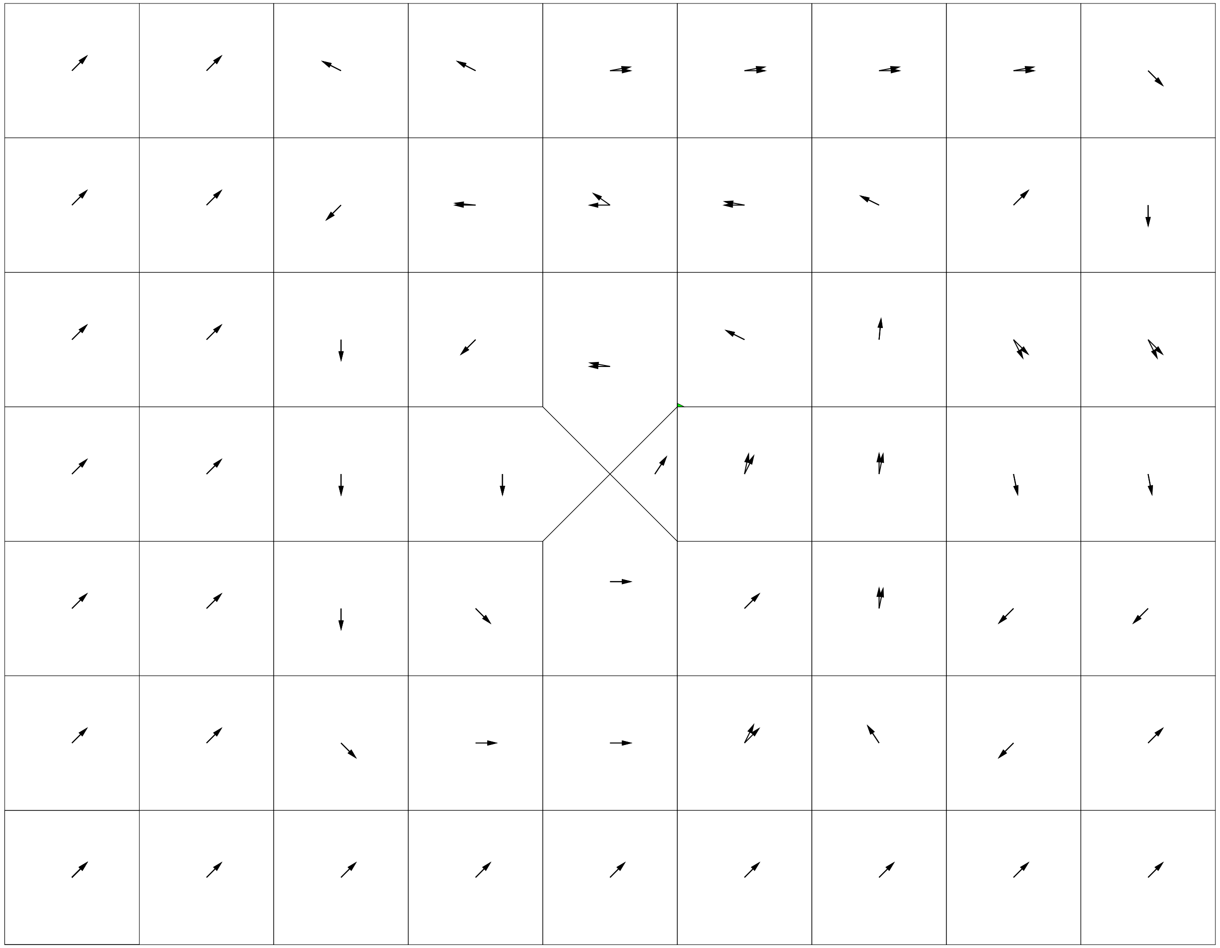

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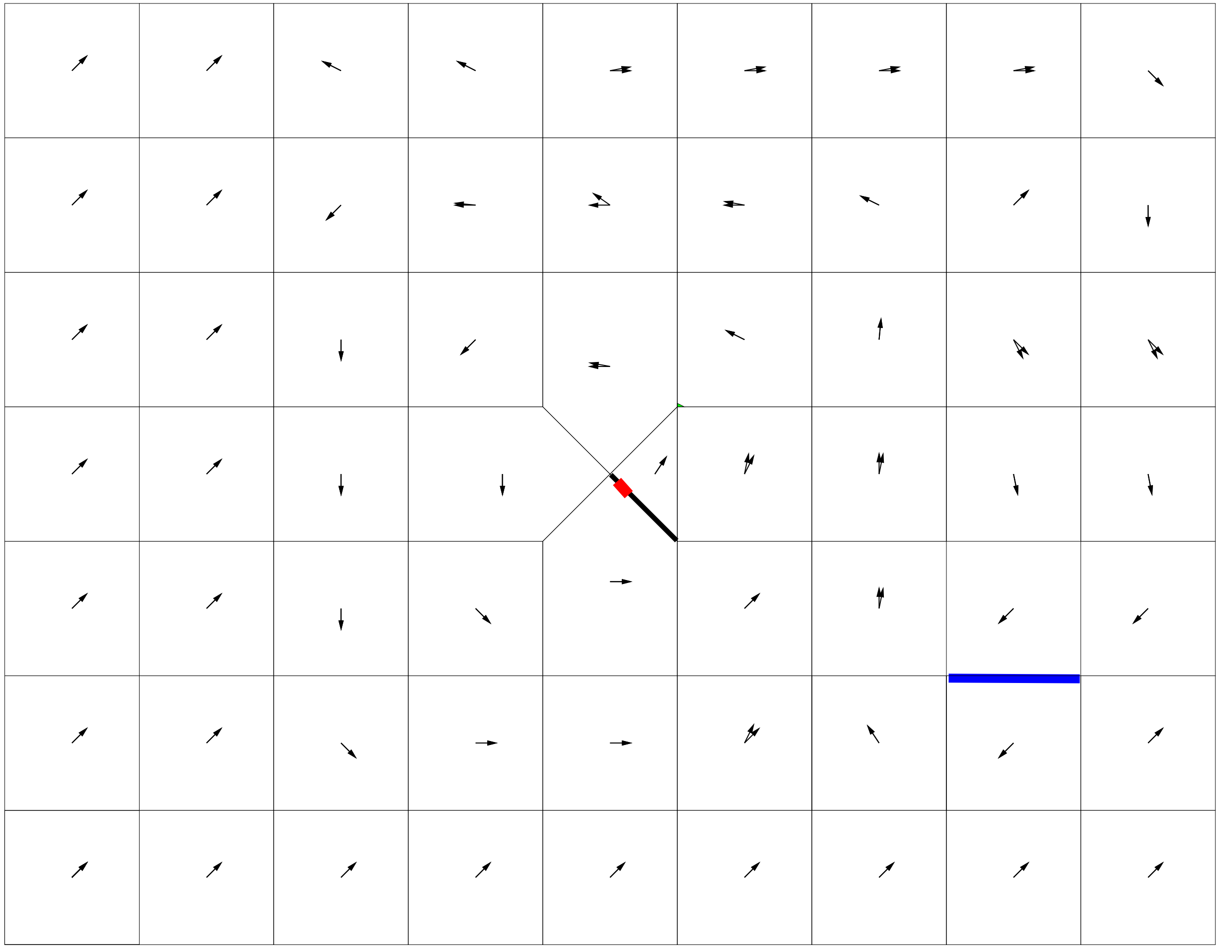
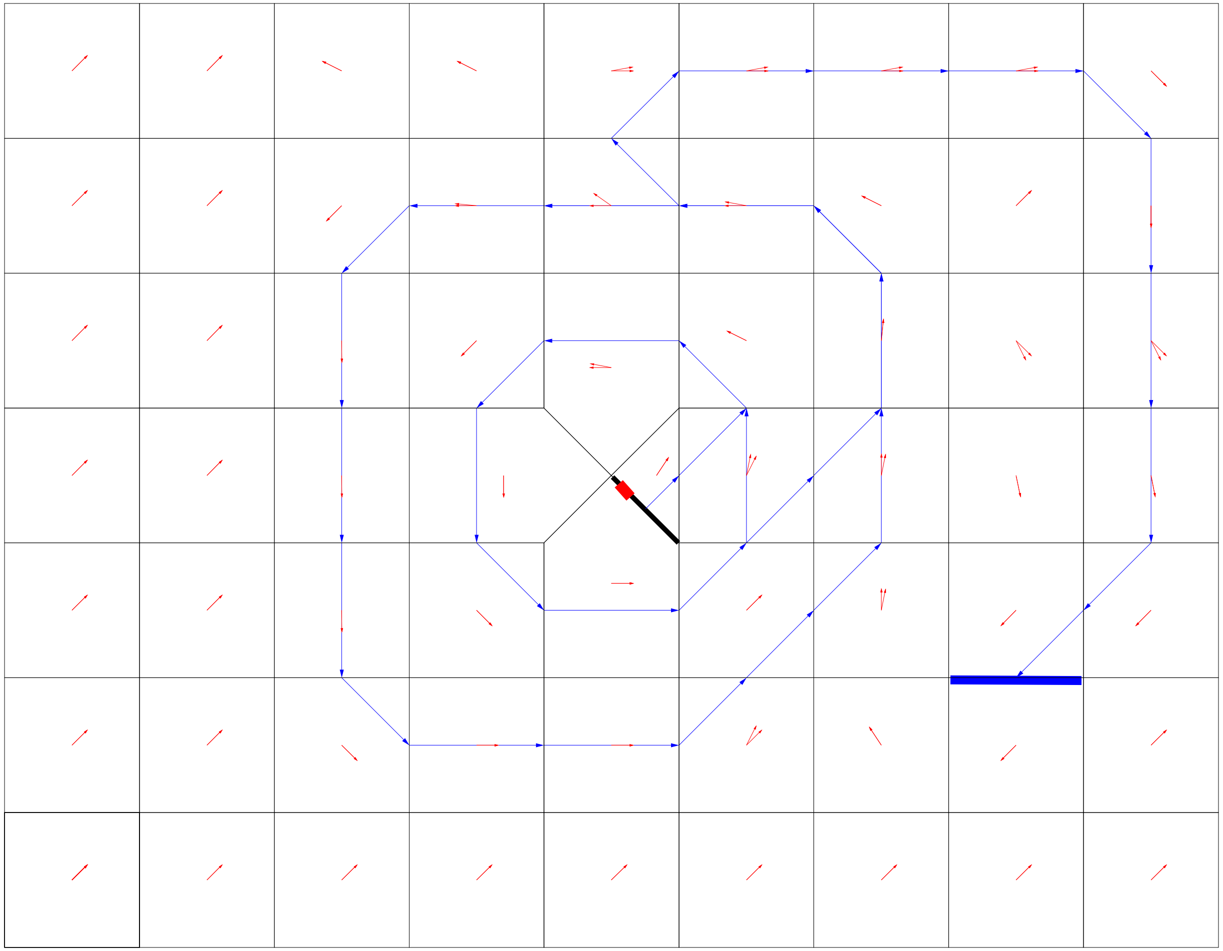


Animate

Implementation: SPeeDI



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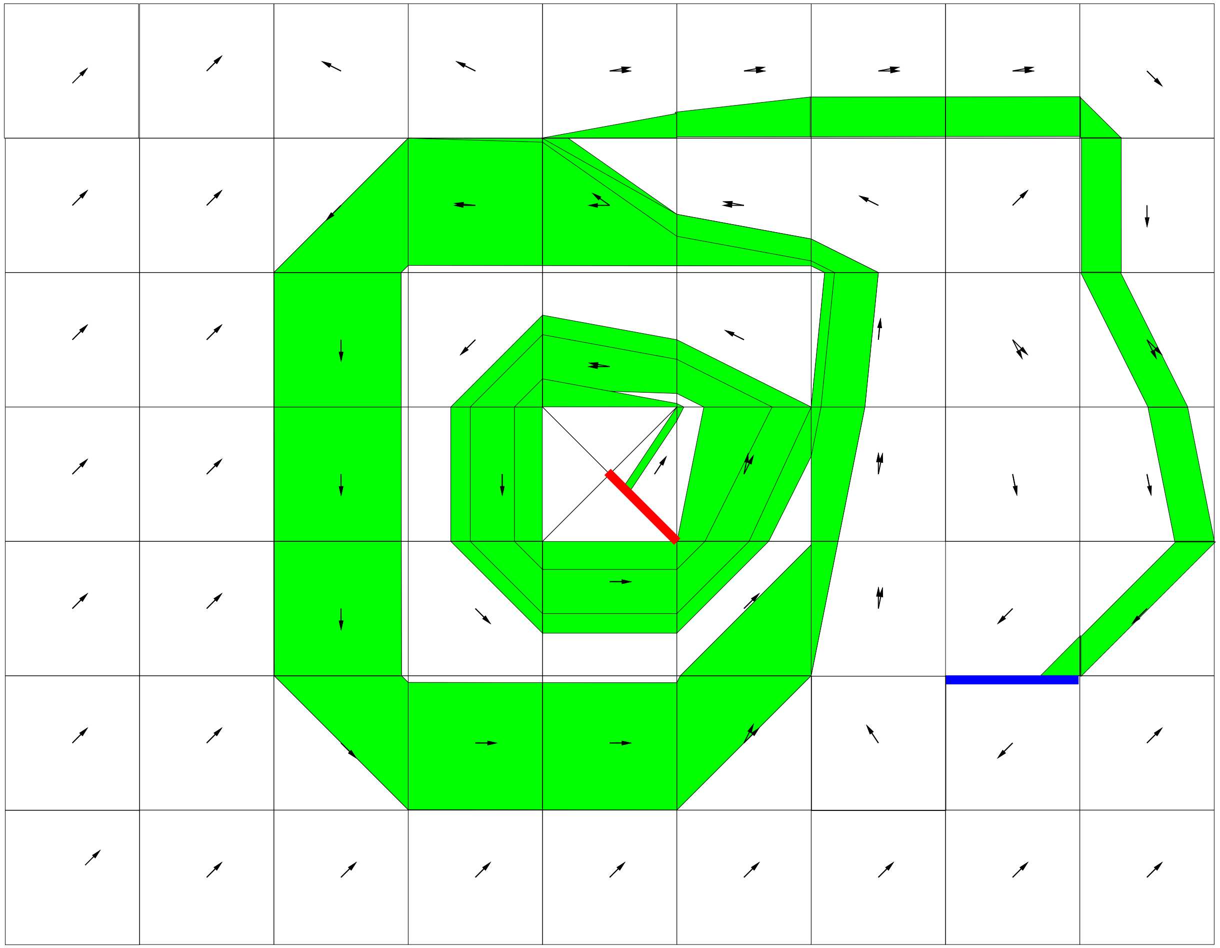

Animate

Phase Portrait of SPDIs

Phase Portrait

Phase Portrait: a picture of important objects of a dynamical system

Phase Portrait

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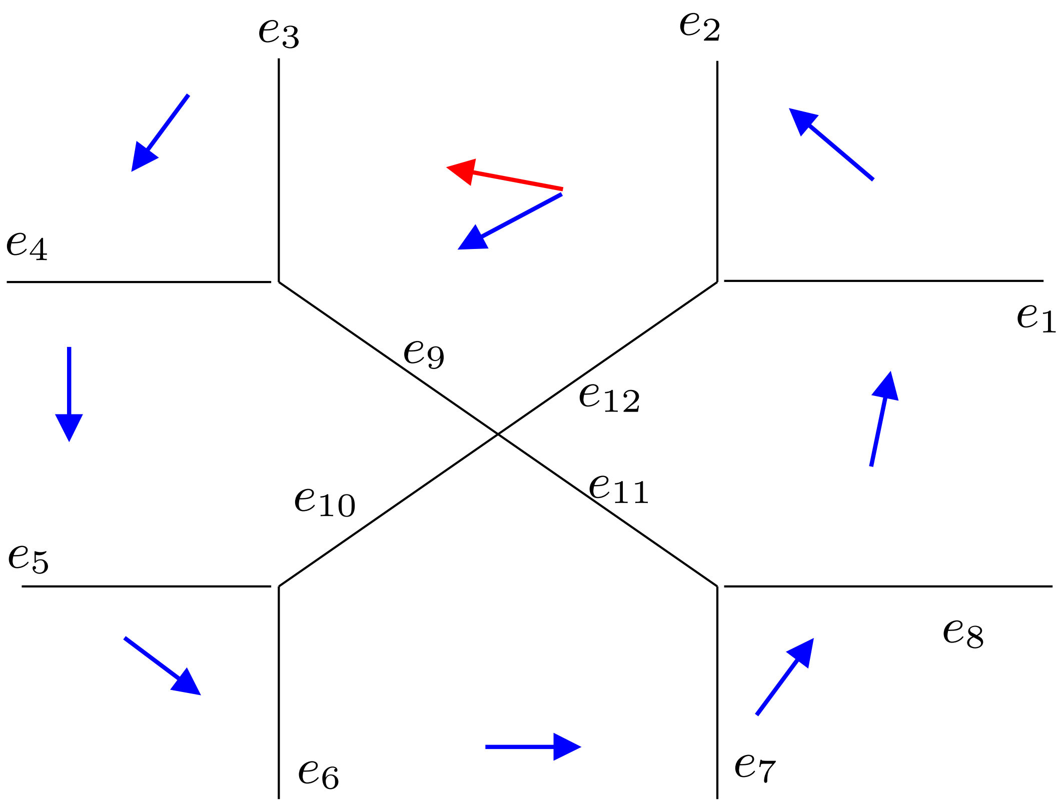
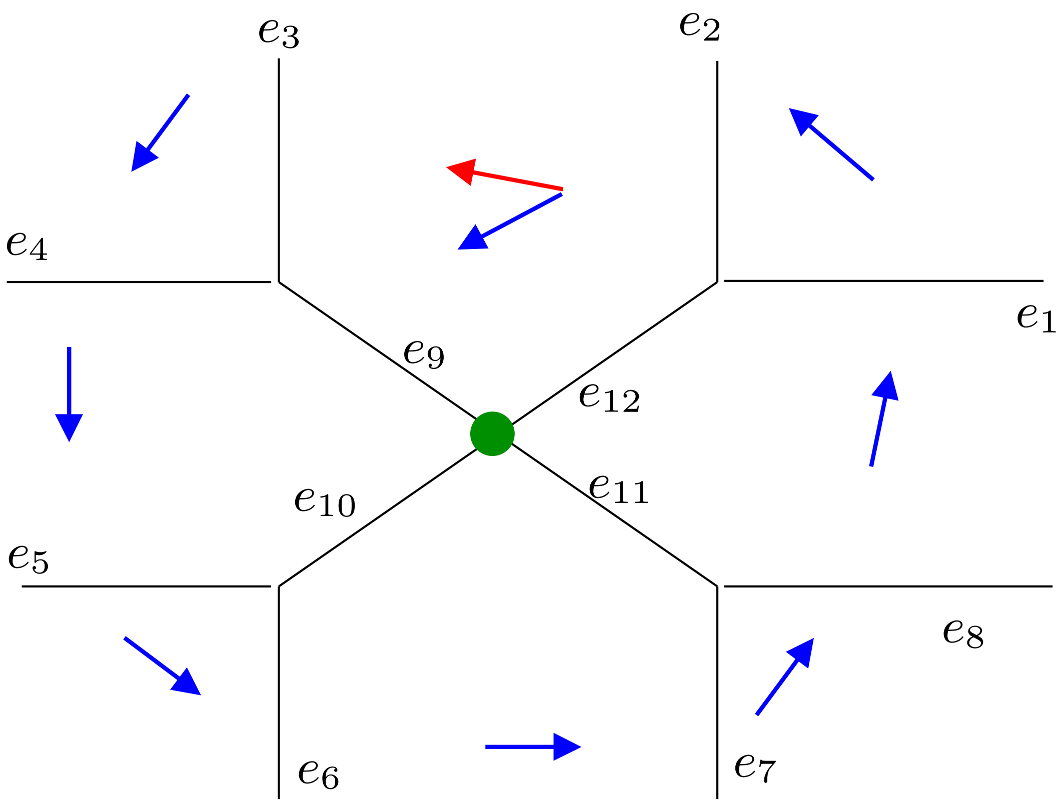


Phase Portrait

Phase Portrait: a picture of important objects of a dynamical system



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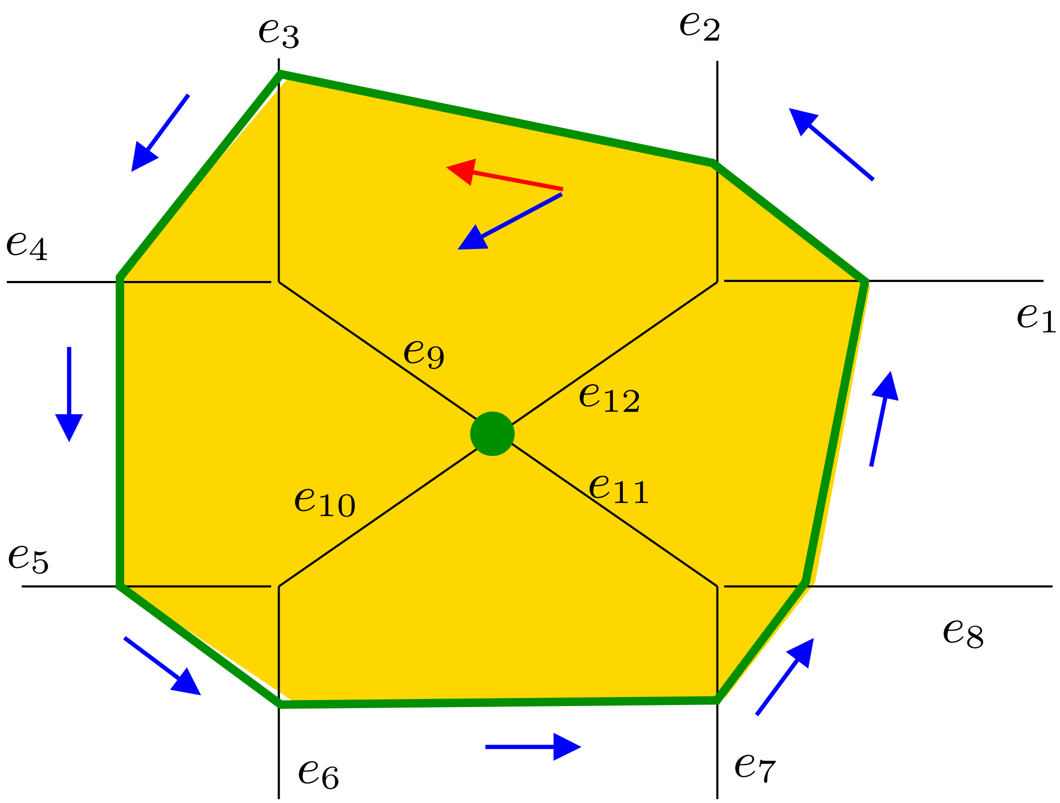

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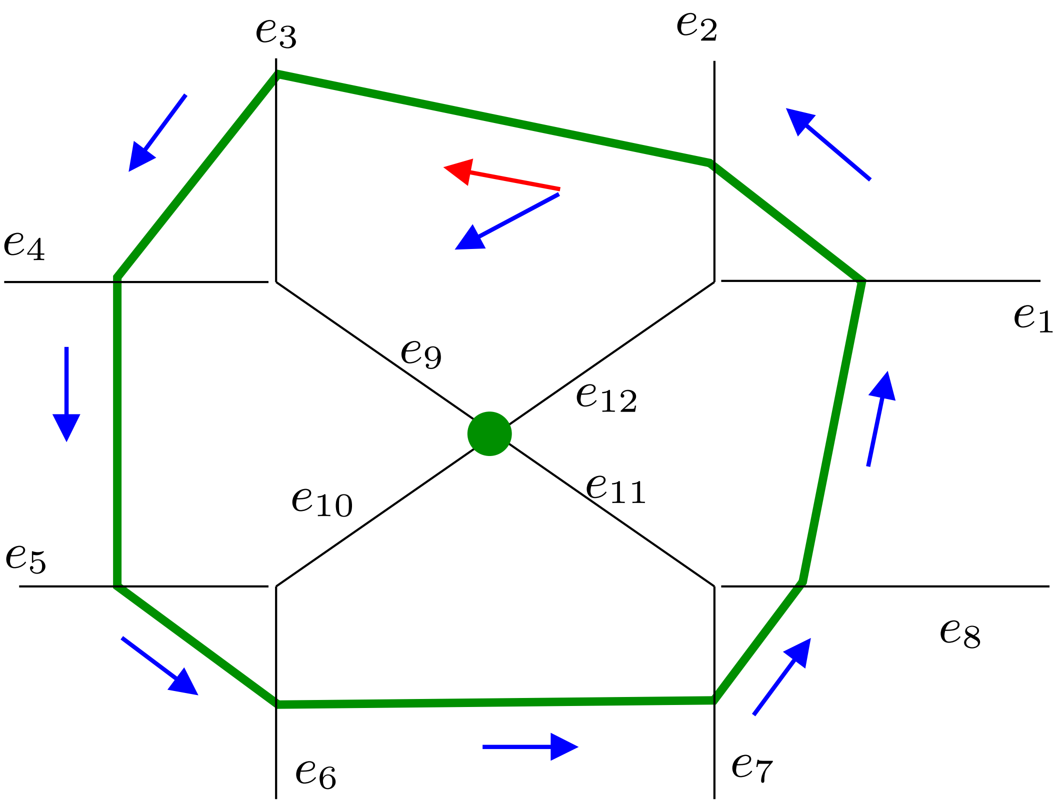


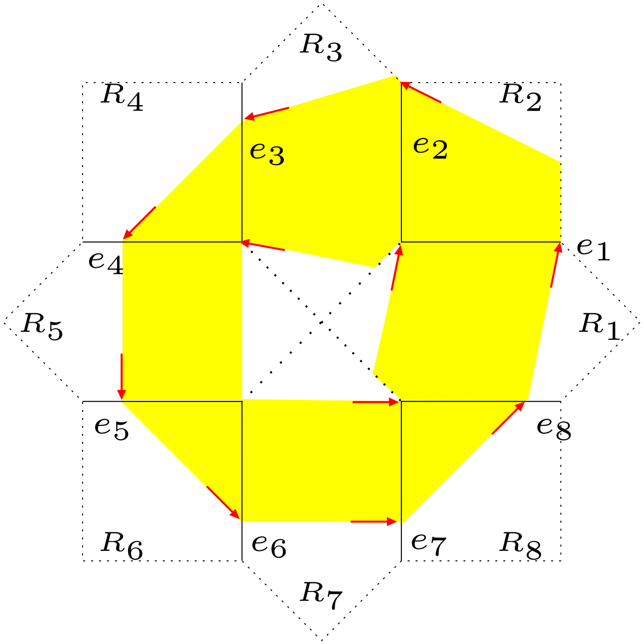
Viability Kernel

$\mathsf{Viab}(\sigma)$: Is the greatest set of initial points of trajectories which can cycle forever in σ

Viability Kernel

Viab(σ): Is the greatest set of initial points of trajectories which can cycle forever in σ

Example: $\sigma = e_1 e_2 \dots e_8 e_1$



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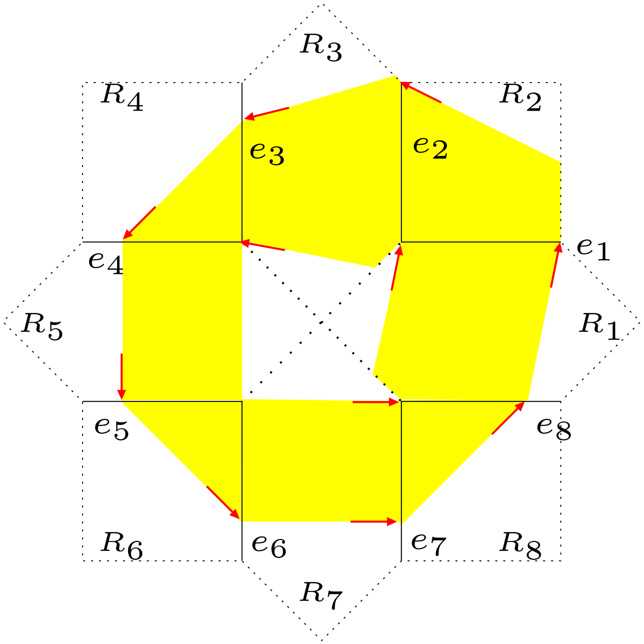

Theorem: $\mathsf{Viab}(\sigma) = \overline{\mathsf{Pre}}_\sigma(\mathsf{Dom}(\mathsf{Succ}_\sigma))$

Controllability Kernel

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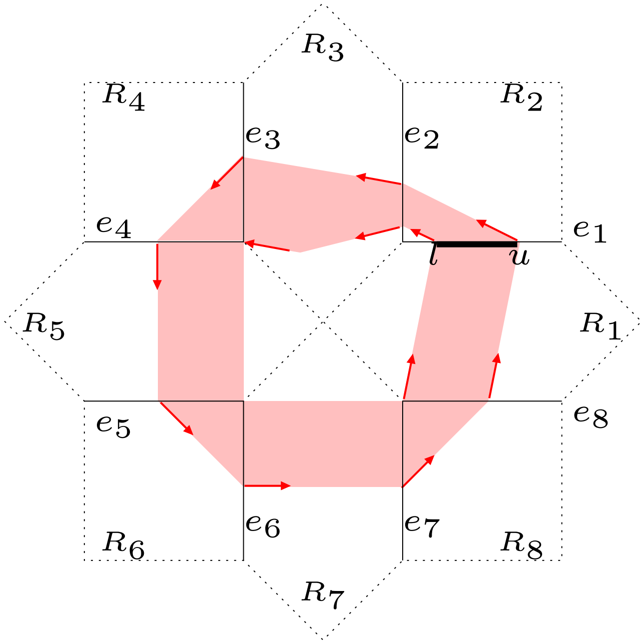
Example: $\sigma = e_1 e_2 \ldots e_8 e_1$



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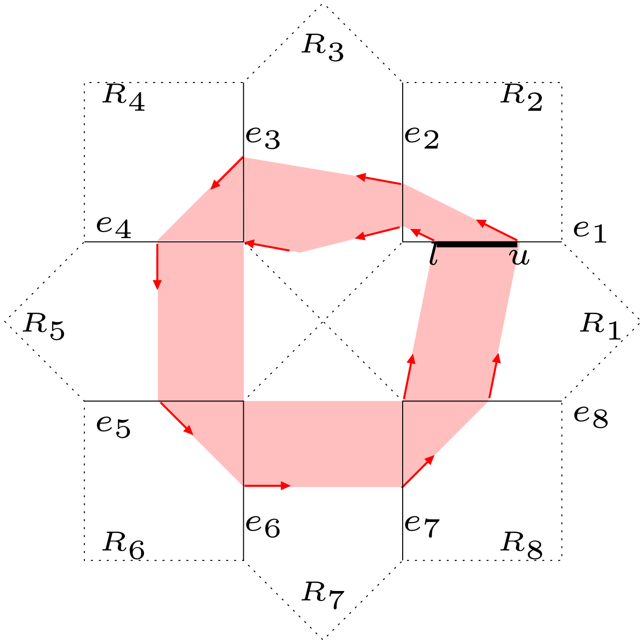


Theorem: $\mathsf{Cntr}(\sigma) = (\overline{\mathsf{Succ}}_\sigma \cap \overline{\mathsf{Pre}}_\sigma)(\mathcal{C}_{\mathcal{D}}(\sigma))$

Viability Kernel

Algorithm: phase portrait for SPDIs

for each simple cycle σ **do**
Compute $\mathsf{Viab}(\sigma)$ (*viability kernel*)
Compute $\mathsf{Cntr}(\sigma)$ (*controllability kernel*)

Viability Kernel

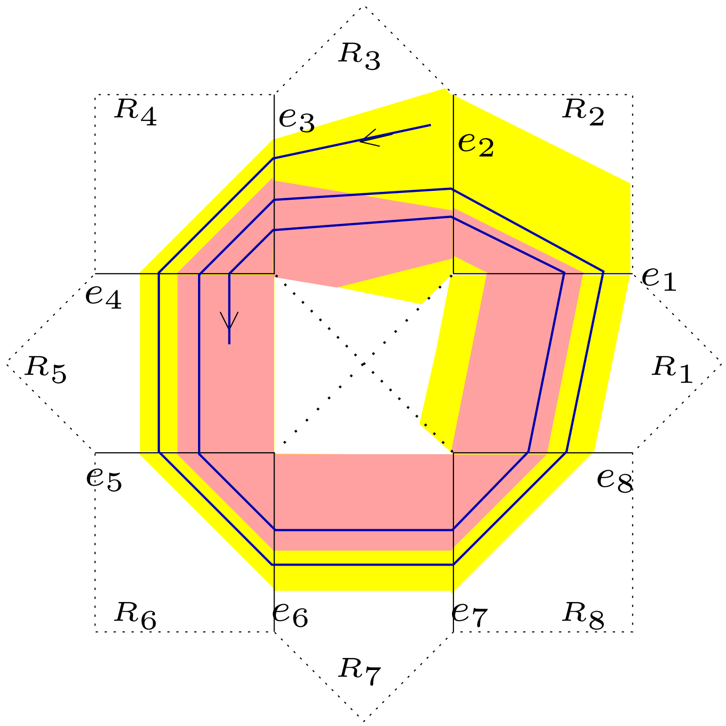
Algorithm: phase portrait for SPDIs

for each simple cycle σ **do**
Compute $\mathsf{Viab}(\sigma)$ (*viability kernel*)
Compute $\mathsf{Cntr}(\sigma)$ (*controllability kernel*)

Both kernels are exactly computed by non-iterative algorithms!

Properties of the Kernels

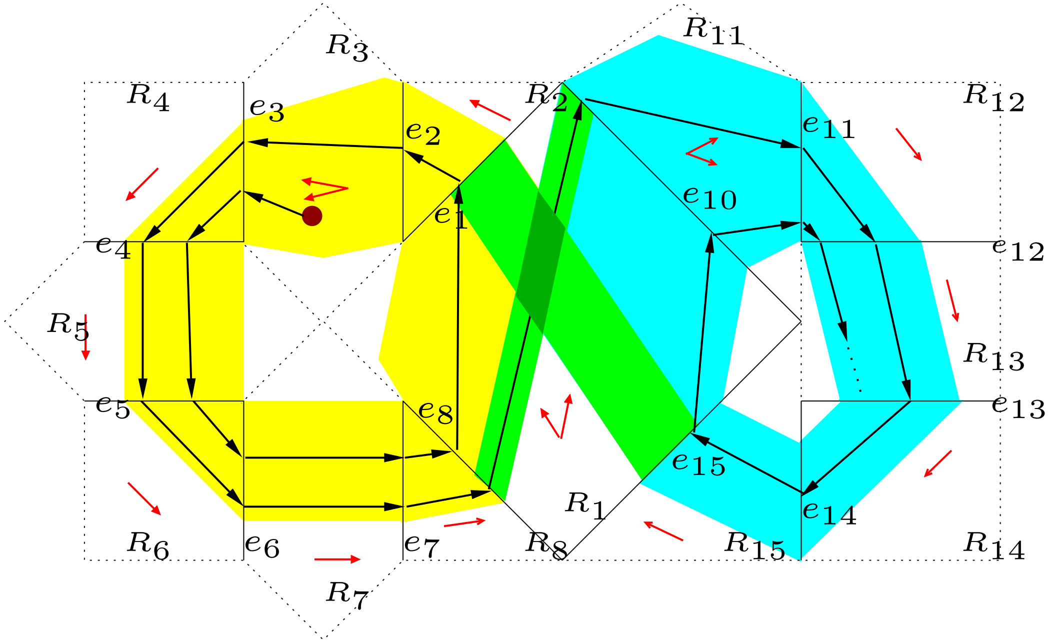
Theorem: Any viable trajectory in σ converges to $\mathsf{Cntr}(K_\sigma)$



- Controllability Kernel: "Weak" analog of limit cycle
- Viability Kernel: Its "local" attraction basin

Convergence Properties

Every trajectory with infinite signature without self-crossings converges to the controllability kernel of some simple edge-cycle



Between Decidable and Undecidable

More complex 2-dim systems

What happens if …

- … we allow jumps?
- … the PCD is on a 2-dim surface/manifold?
- … ?

More complex 2-dim systems

What happens if . . .

- . . . we allow jumps?
- . . . the PCD is on a 2-dim surface/manifold?
- . . . ?

Answer: Reachability is equivalent to a well known open problem

Our Reference Model

1-dim Piecewise Affine Maps (PAMs):

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_i x + b_i$ for $x \in I_i$

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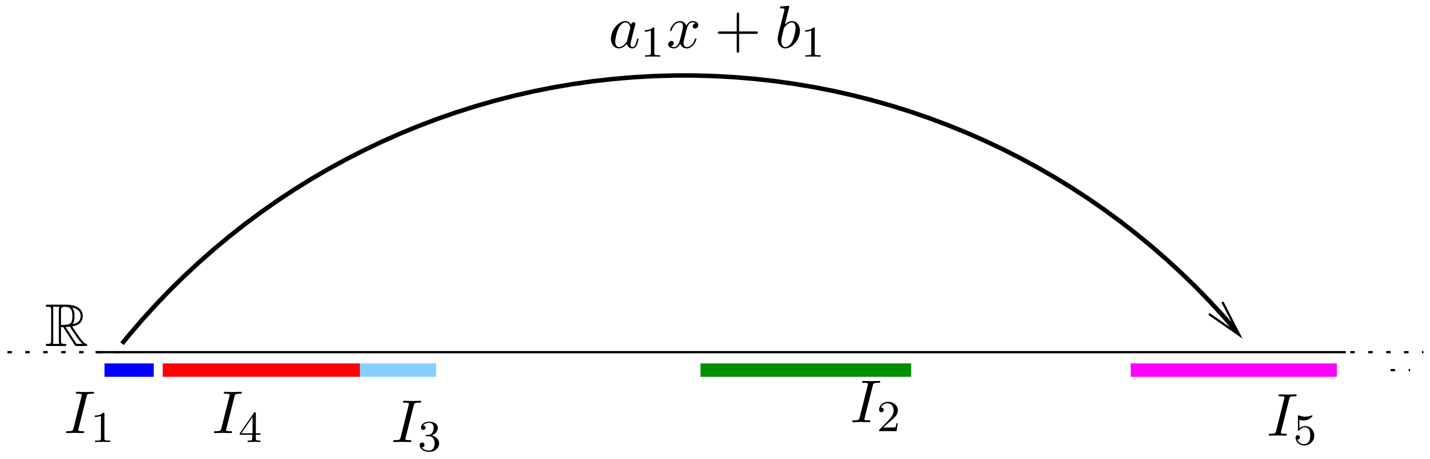

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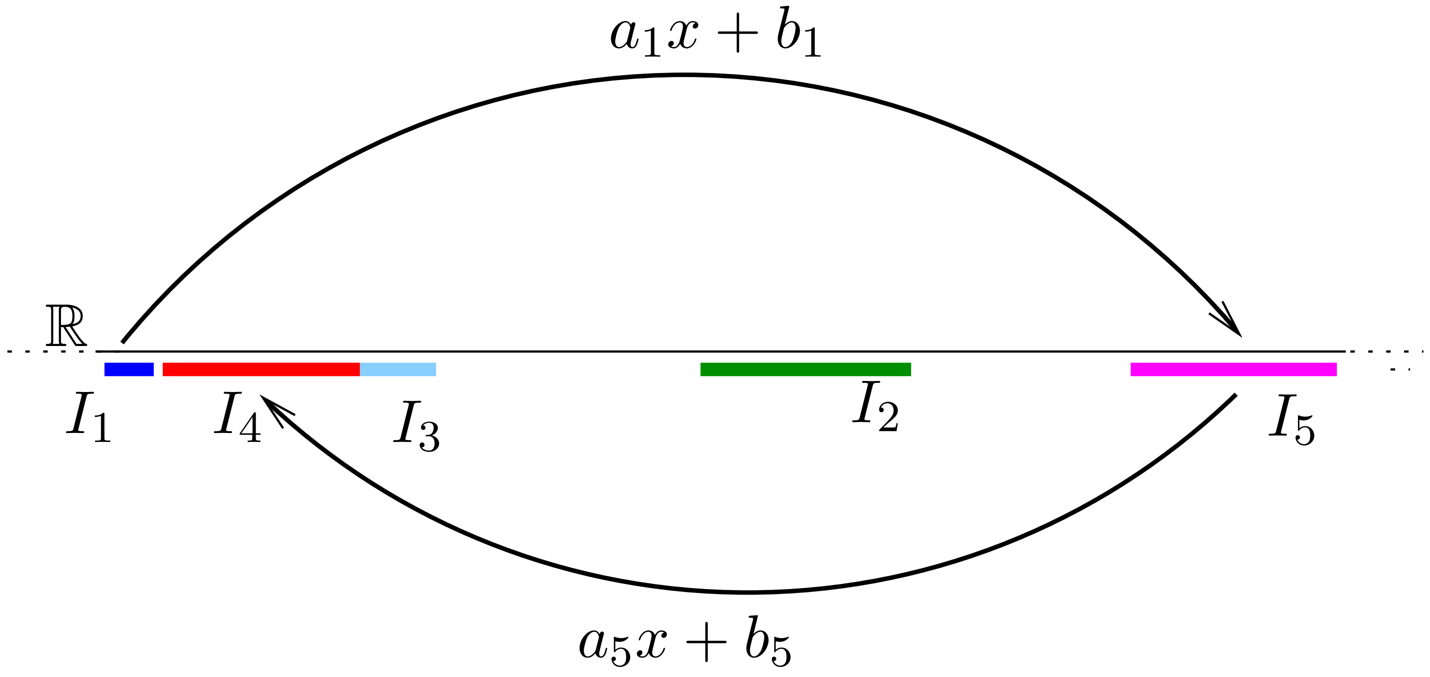

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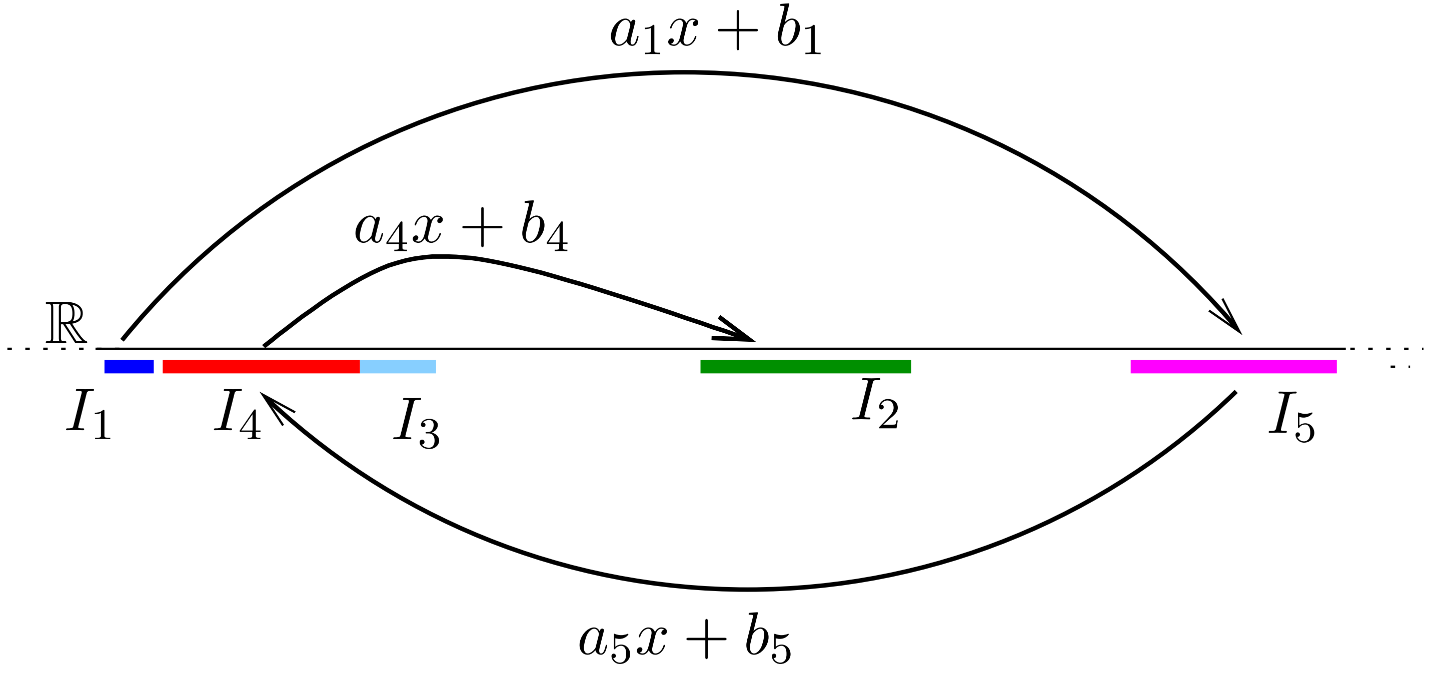

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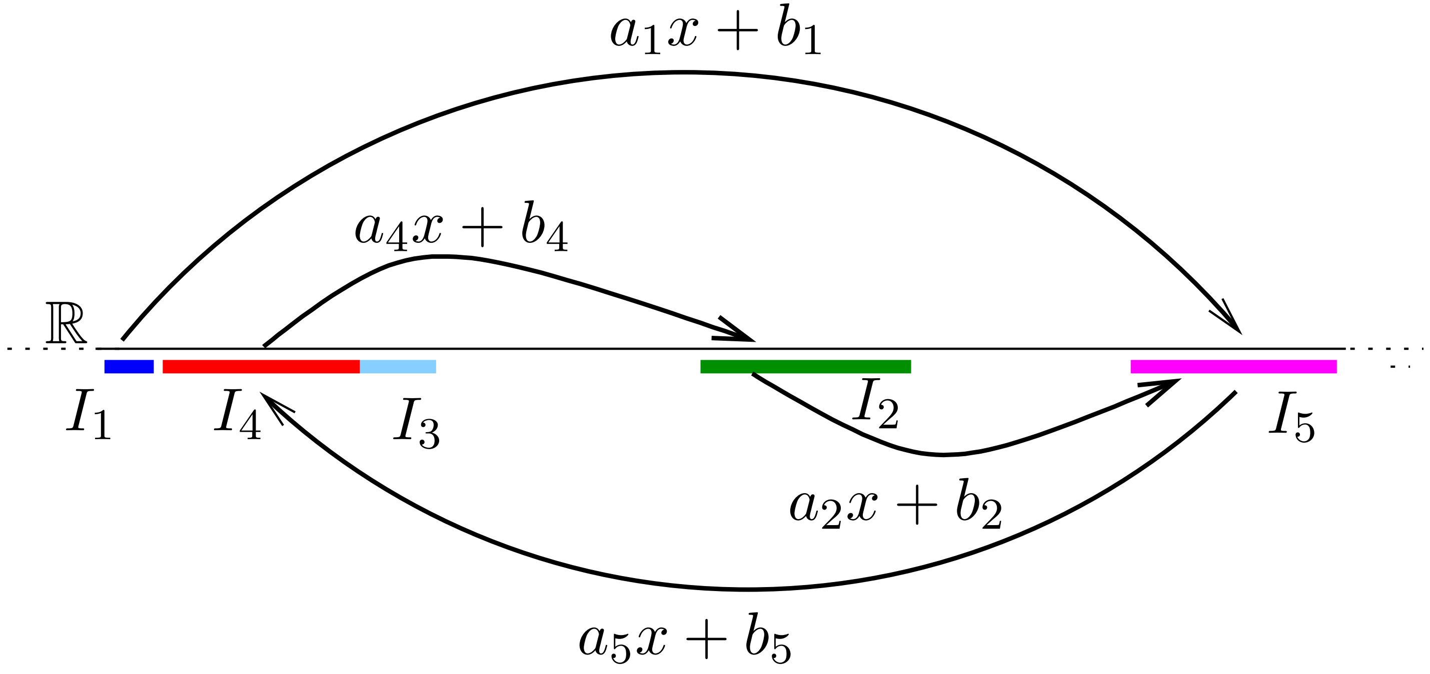

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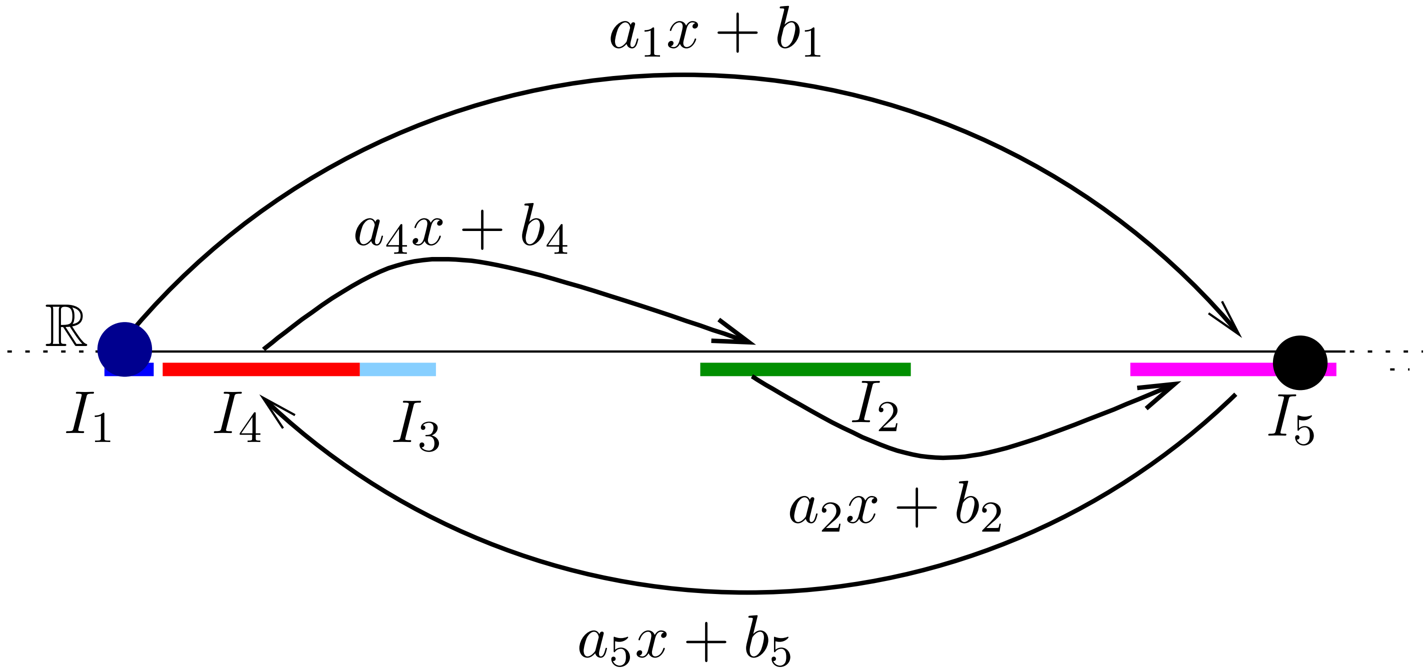

Reachability?

Our Reference Model

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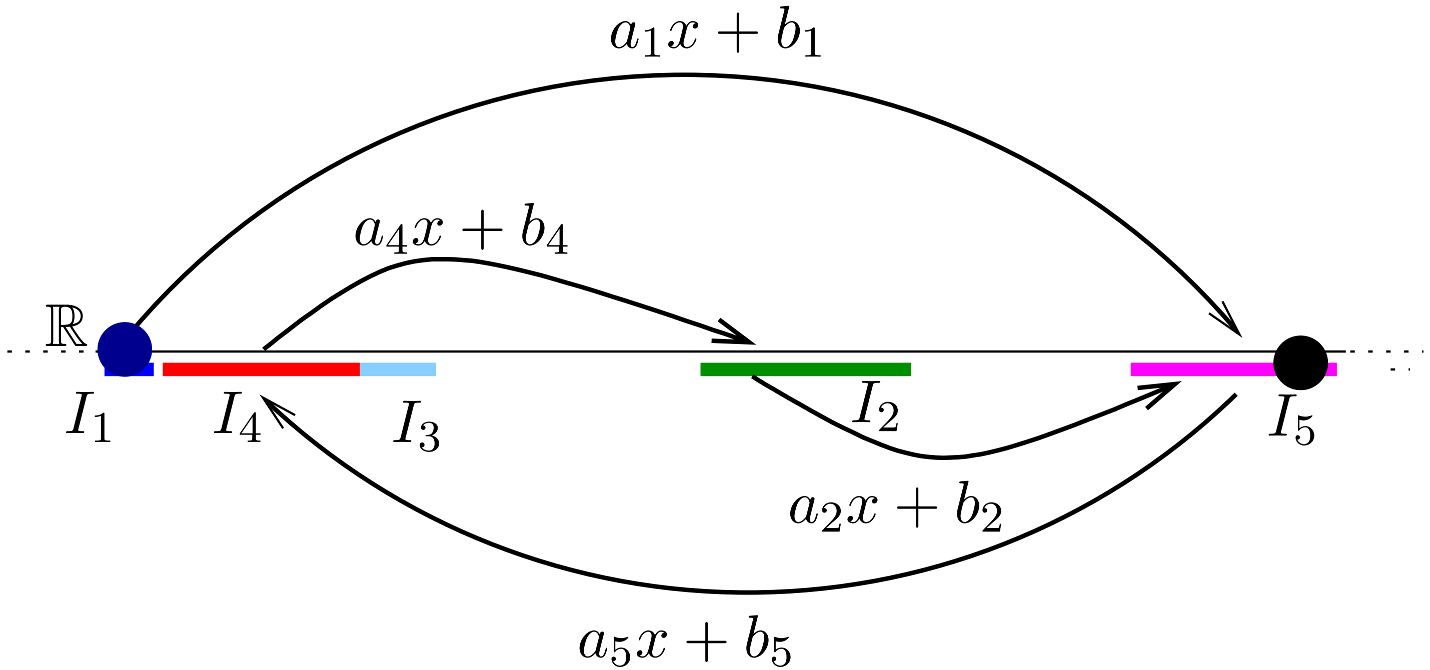
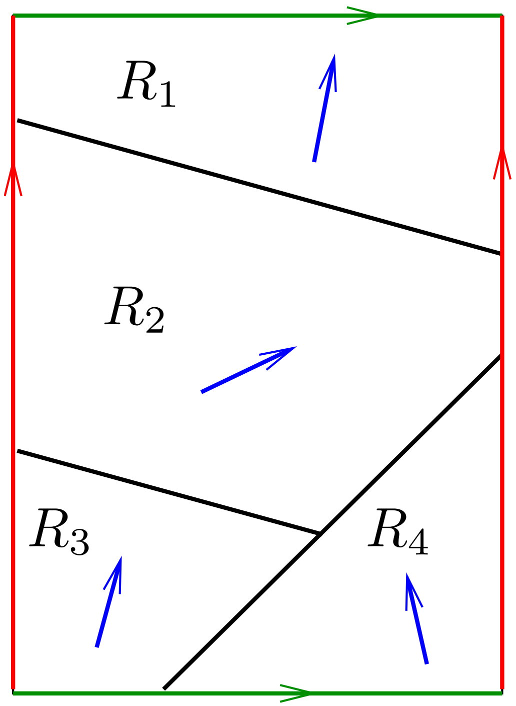


Reachability? Open problem!

PCD on 2-dim manifolds (PCD_{2m})

Example: Torus



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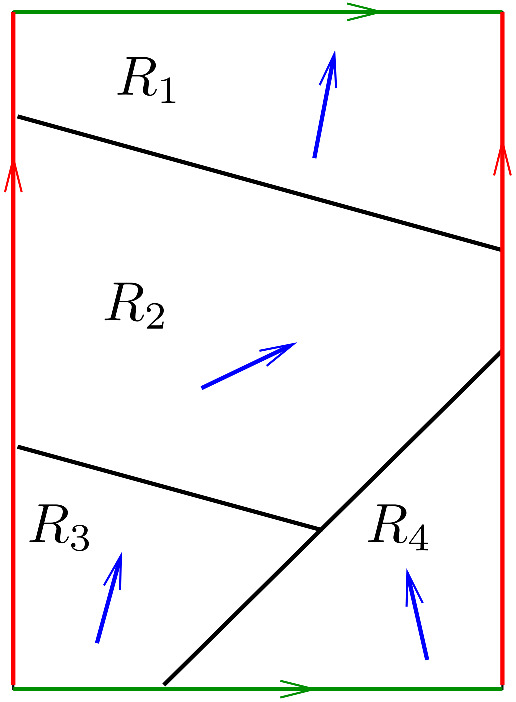

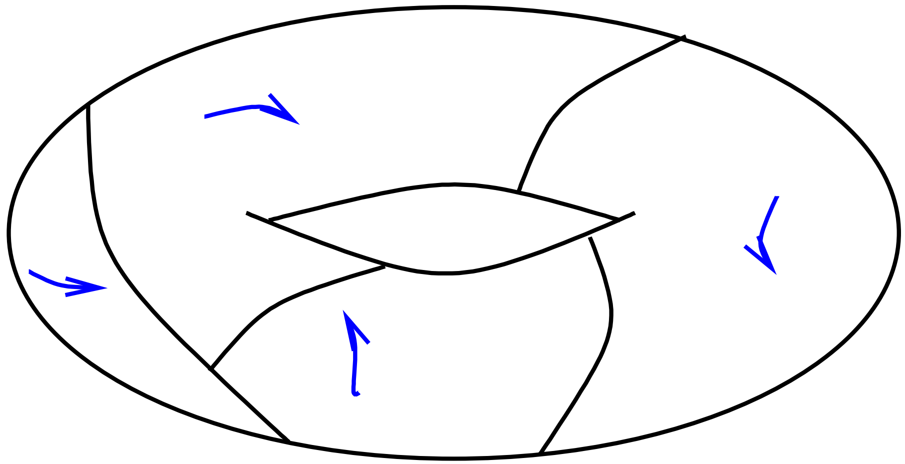

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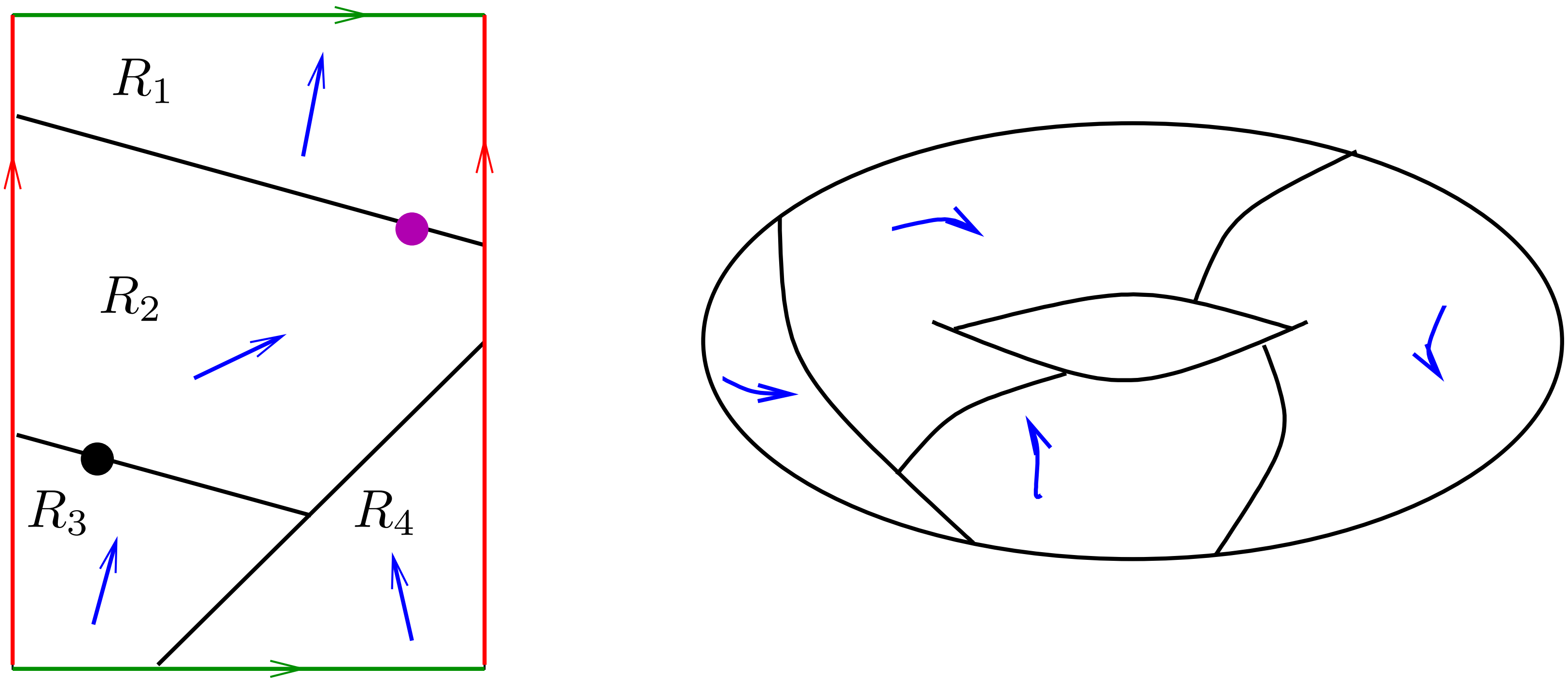

Reachability?

PCD on 2-dim manifolds (PCD_{2m})

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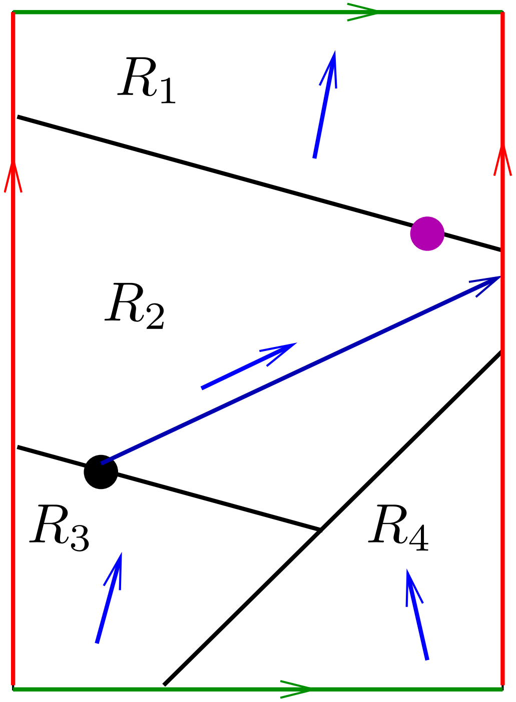

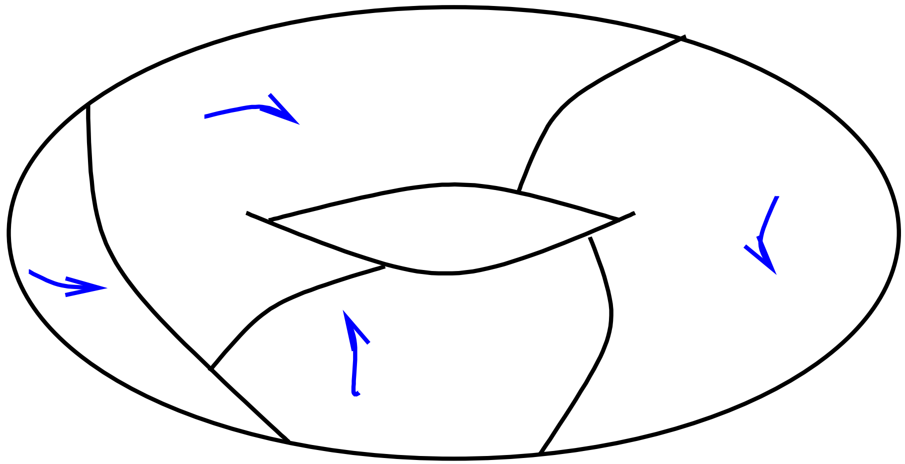

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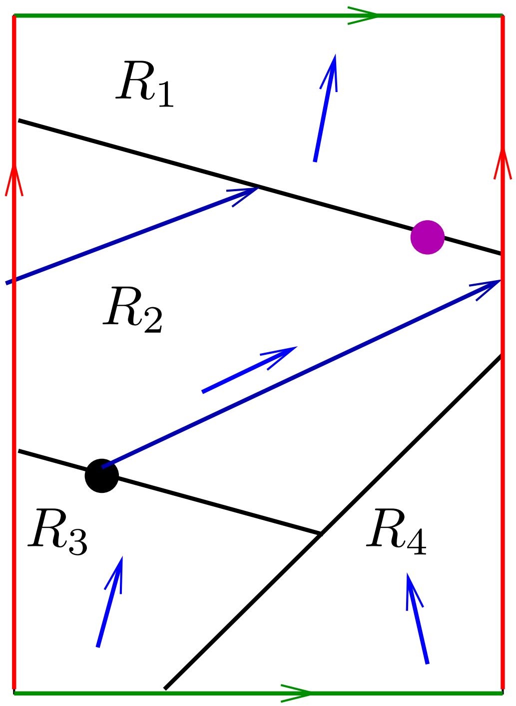

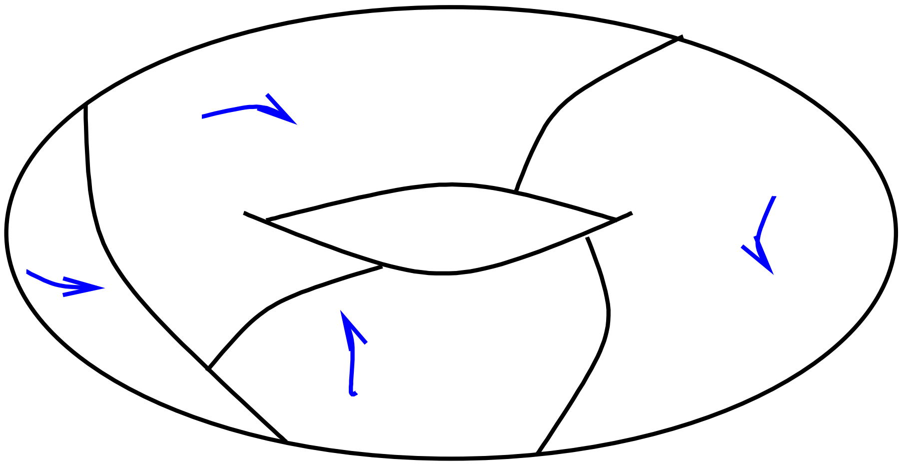

Reachability?

PCD on 2-dim manifolds ($\mathrm{PCD_{2m}}$)

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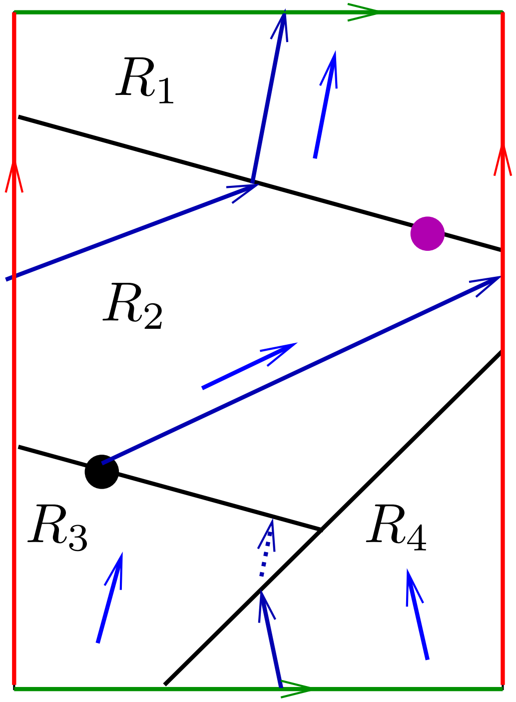

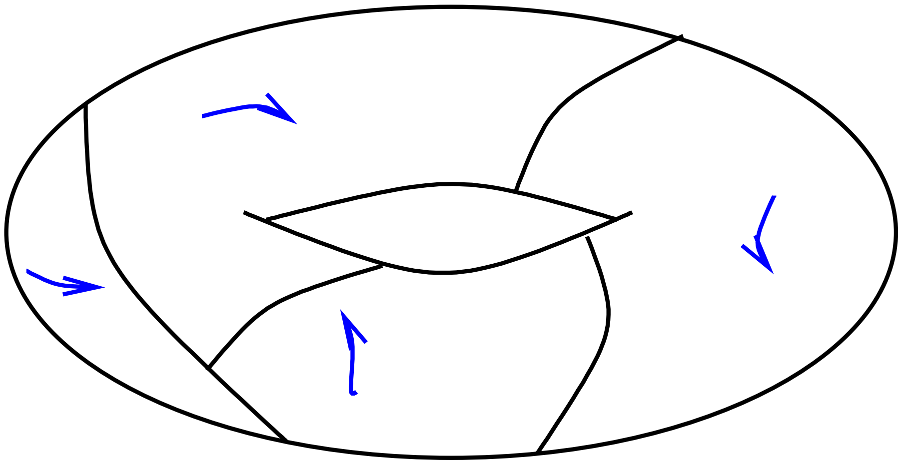

Reachability?

PCD on 2-dim manifolds (PCD_{2m})

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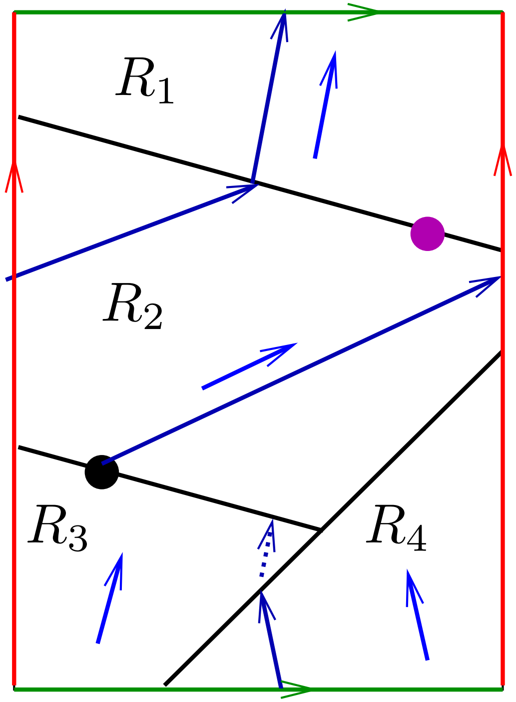
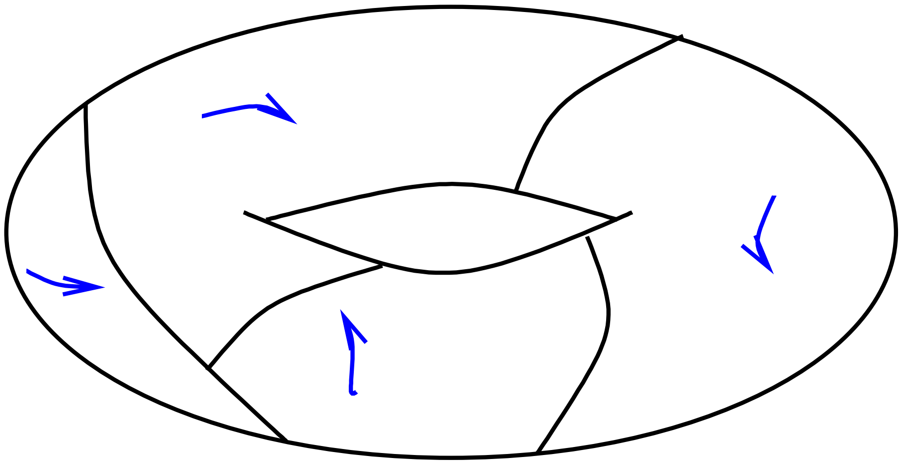
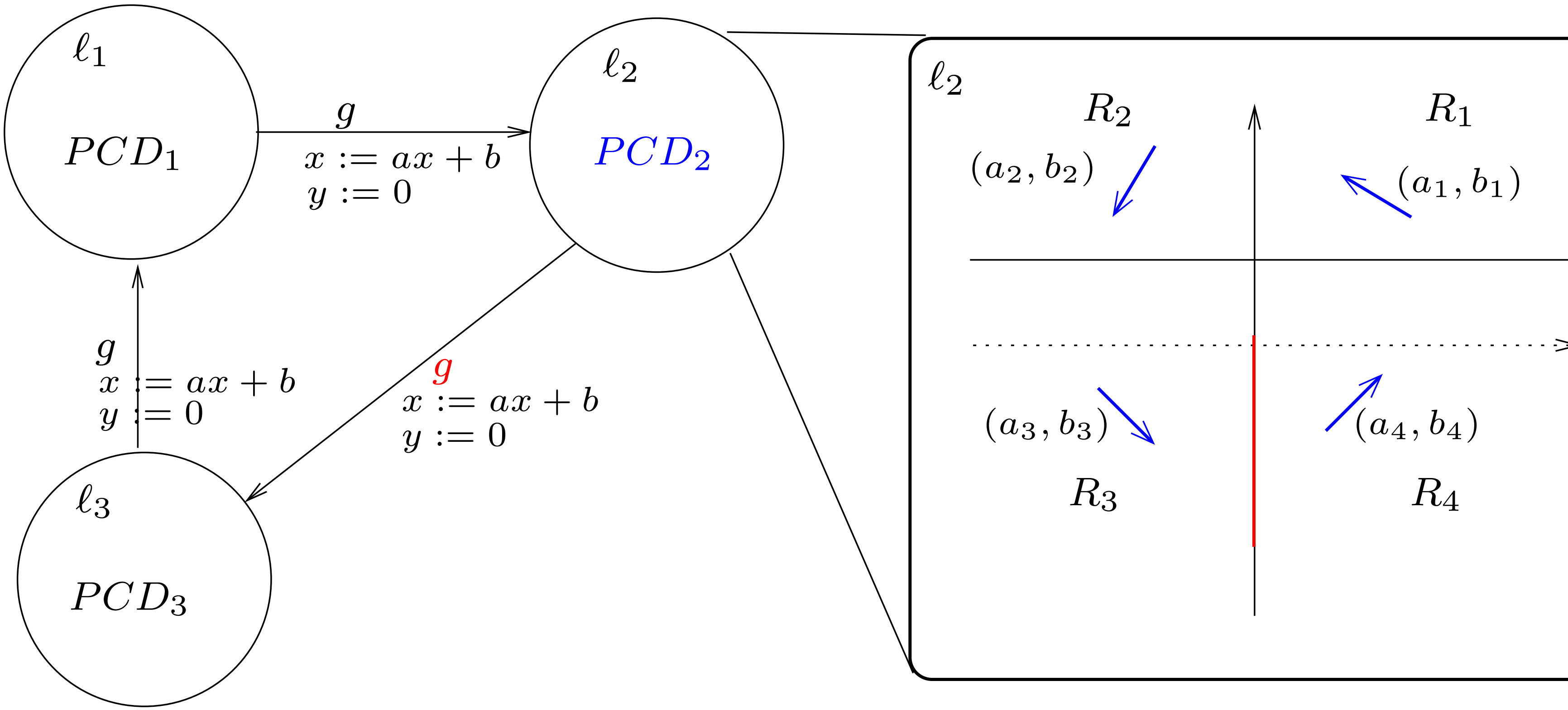




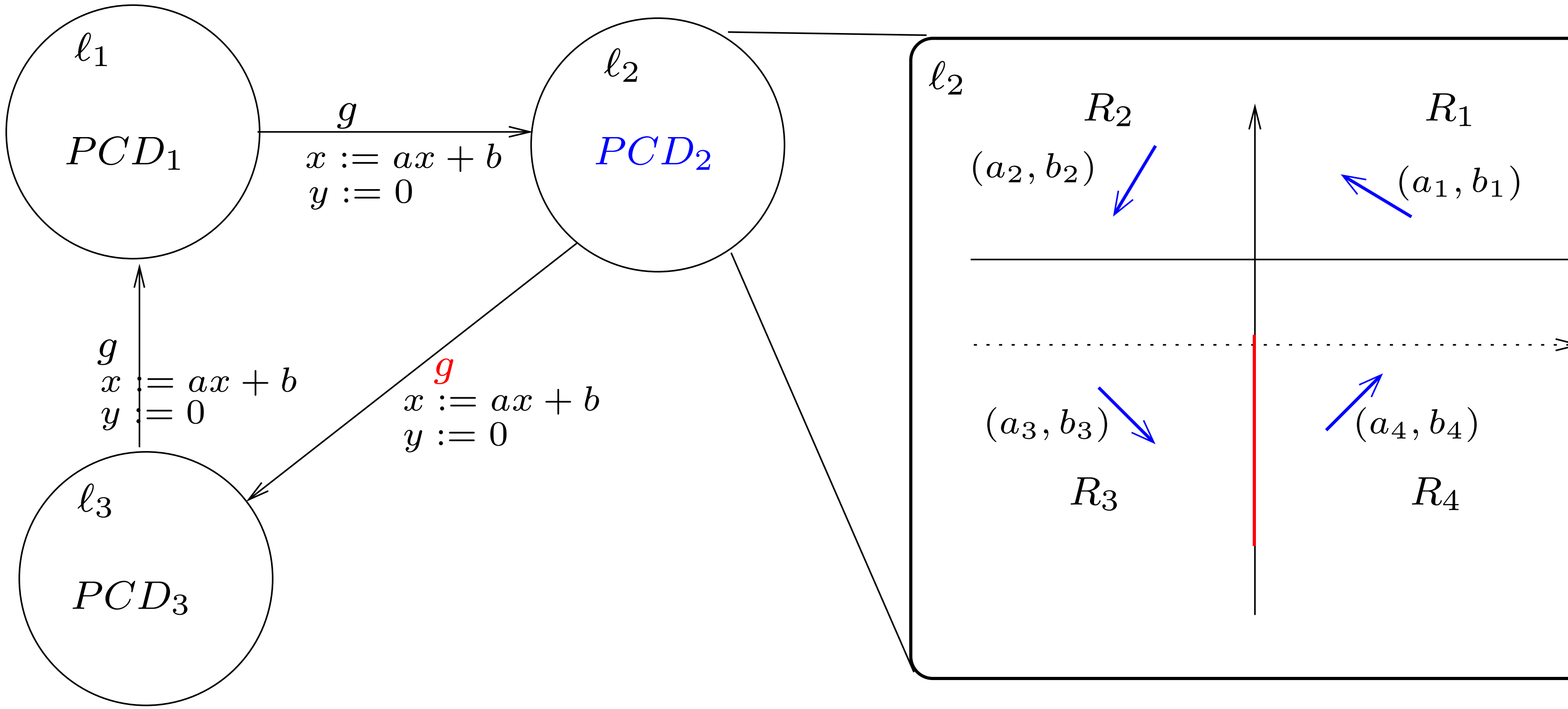
Reachability?

Theorem: $\mathrm{PCD}_{2m} \equiv \mathrm{PAM}$

Hierarchical PCDs (HPCD)



Hierarchical PCDs (HPCD)



Reachability?

Theorem: HPCD $\equiv$ PAM

Undecidable 2-dim Systems

Undecidability Results

- HPCDs with One Counter (HPCD_{1c})
- HPCDs with Infinite Partition (HPCD_{∞})
- Origin-dependent rate HPCDs (HPCD_{x})

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Undecidability Results

- HPCDs with One Counter (HPCD_{1c})
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- Origin-dependent rate HPCDs (HPCD_{x})

Reachability? UNDECIDABLE!

Theorem:

HPCD_{1c}, HPCD_{∞} and HPCD_{x}

simulate

Turing machines

Summary of Results

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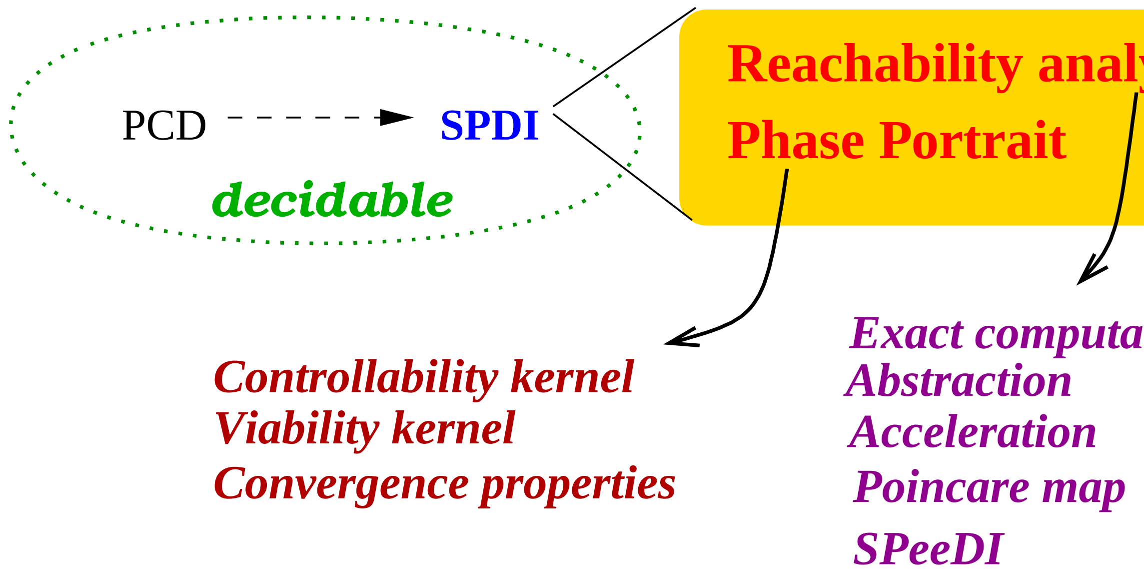
PCD - - - - - - -► **SPDI**

A - - - -► B "A is a particular case of B"

Summary of Results



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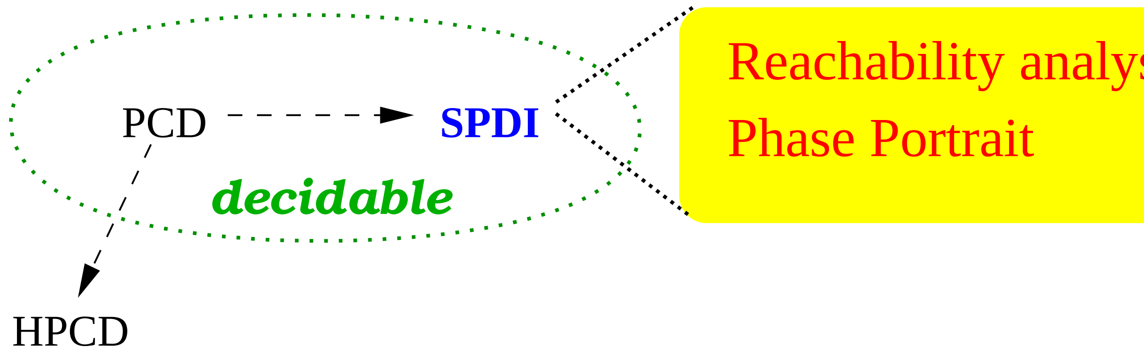

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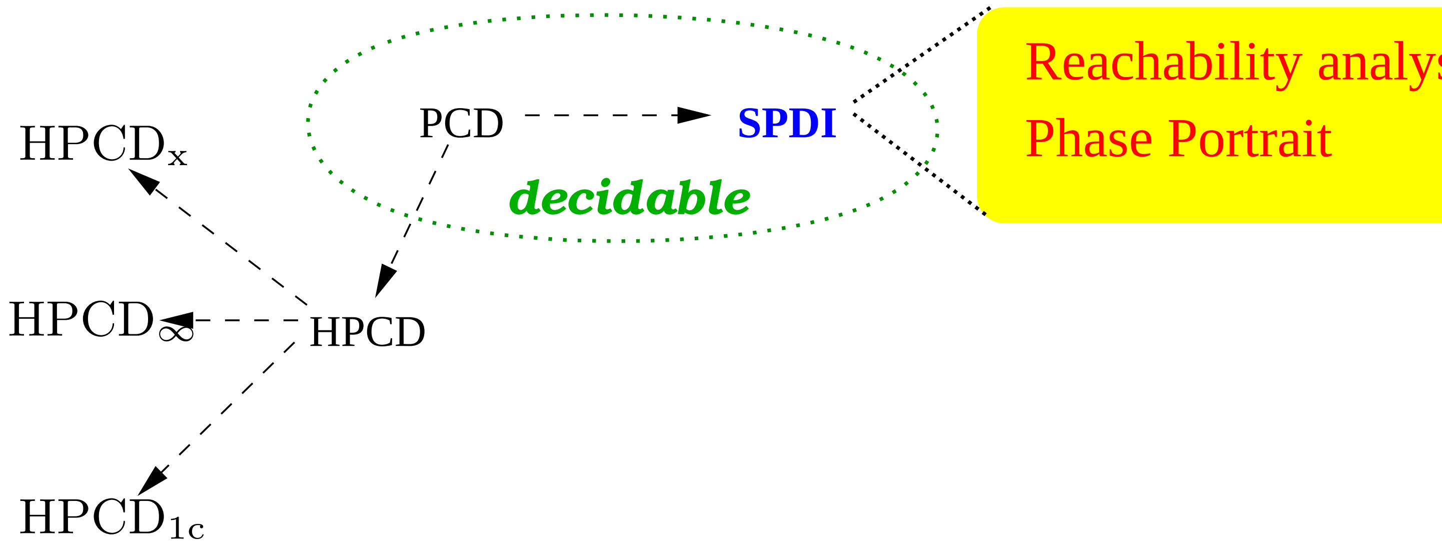

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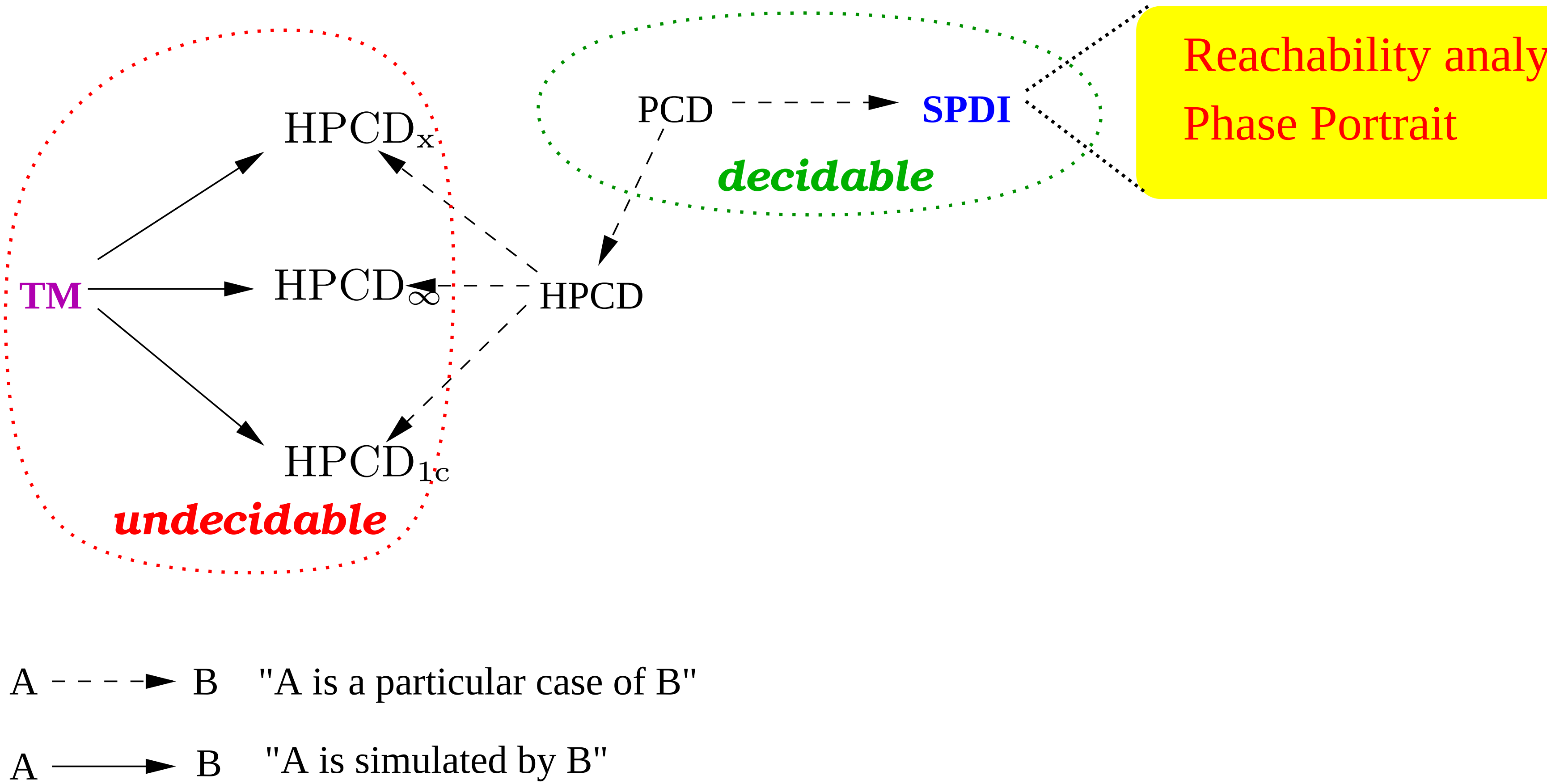

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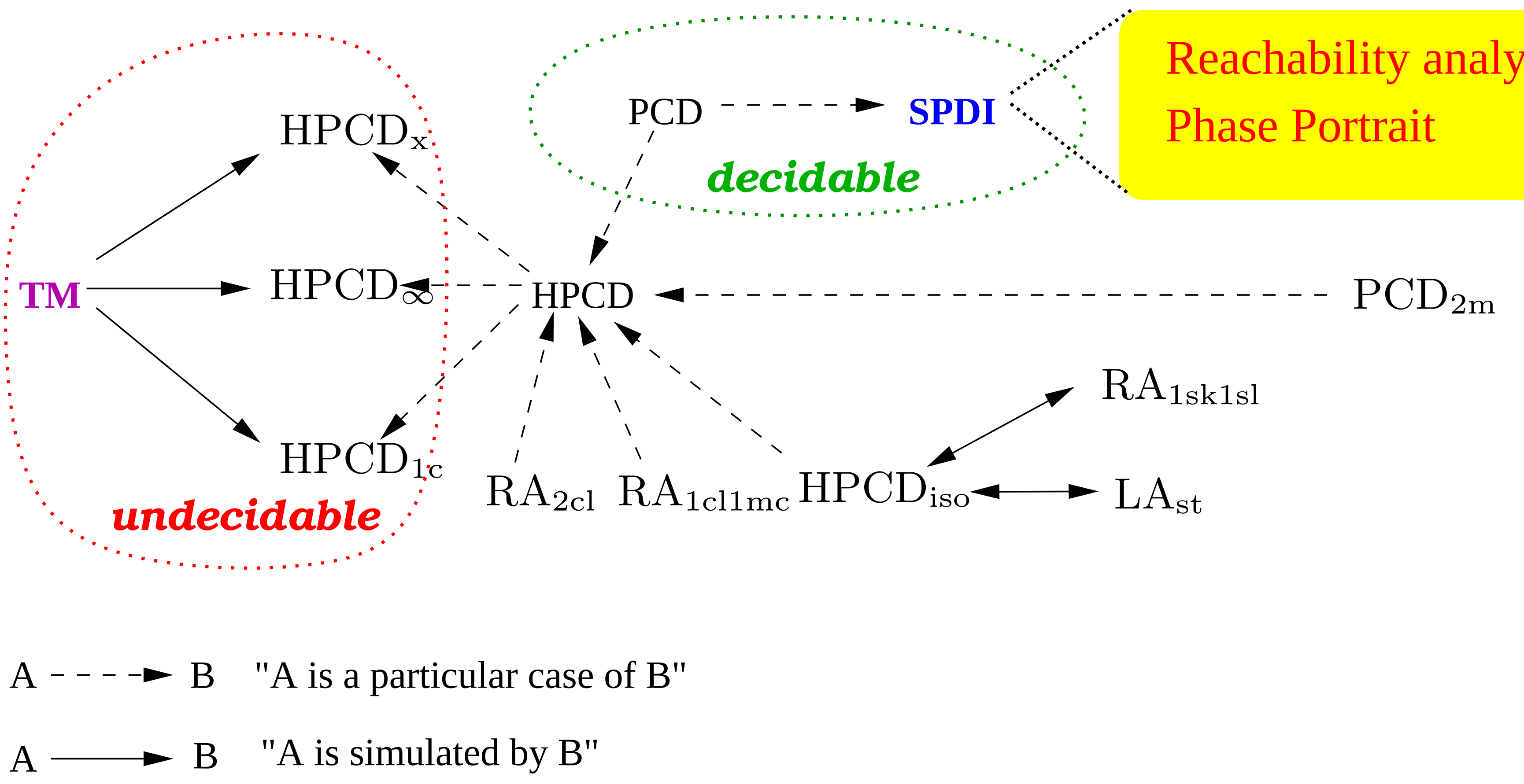
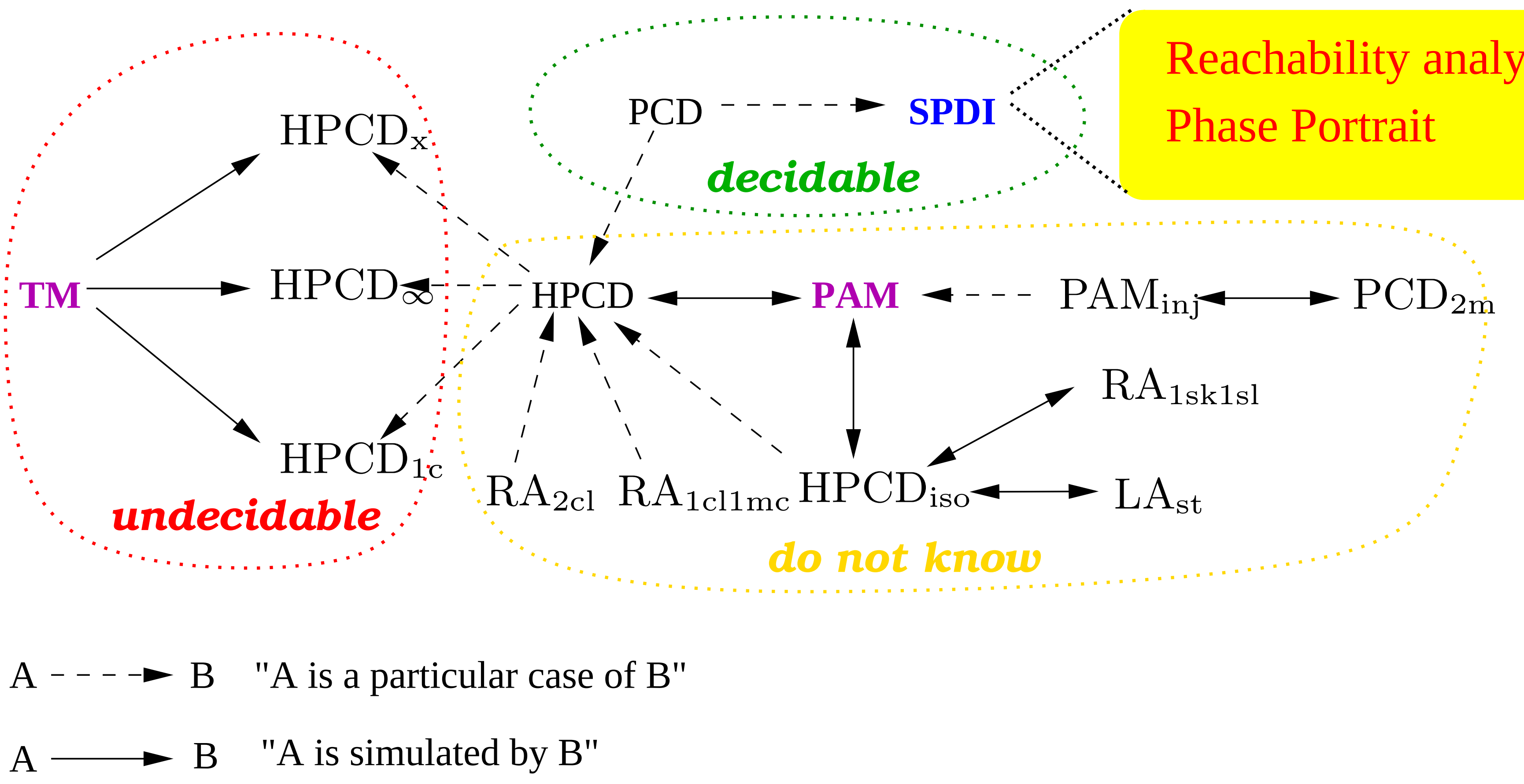


Summary of Results



Perspectives

- SPDI to approximate non-linear differential equations
- Conditions for decidability of PCDs on 2-dim manifolds
- Application of the *geometric* method to higher dimensions
- Extension of SPeeDI: algorithm for viability and controllability kernels
- SPeeDI: “Topological” optimizations

Merci!

Gracias!

Obrigado!

(Brasil Penta-Campeão!)

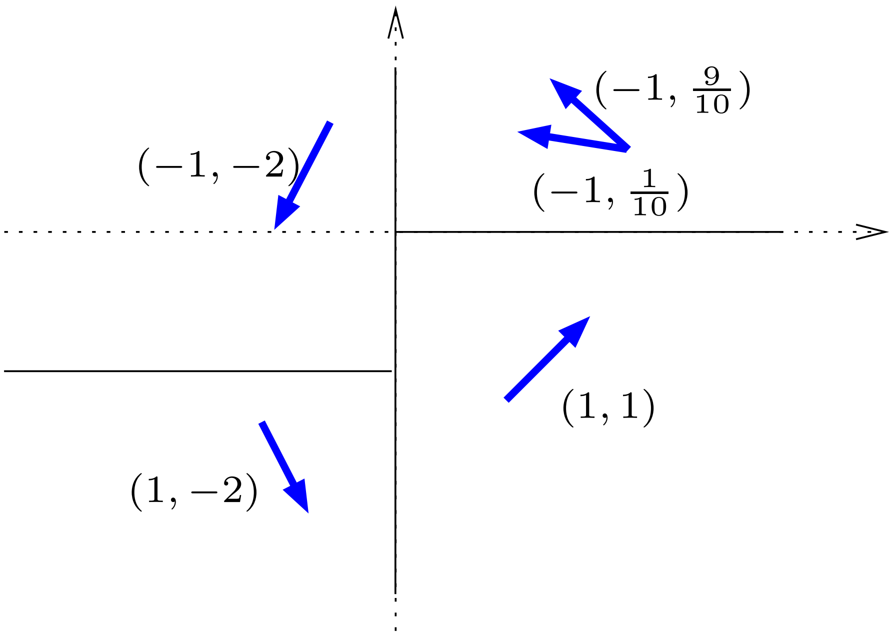
Thank you!

Theorem de Poincaré-Bendixson

A non-empty compact limit set of C^1 planar dynamical system that contains no equilibrium points is a close orbit.

Comparison with HyTech

Example:



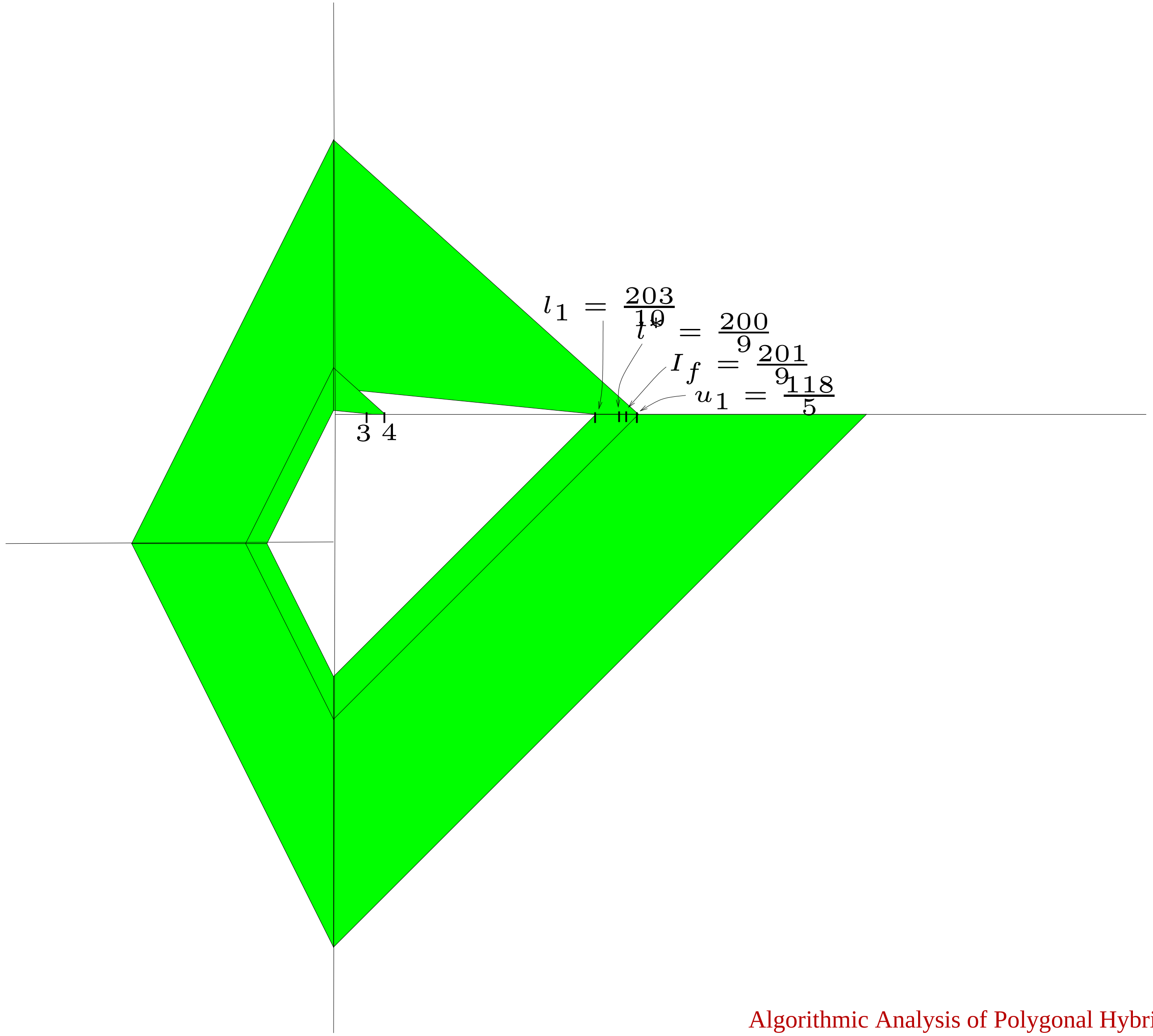
Fixpoint: $I^* = (\frac{200}{9}; 200)$

Comparison with HyTech

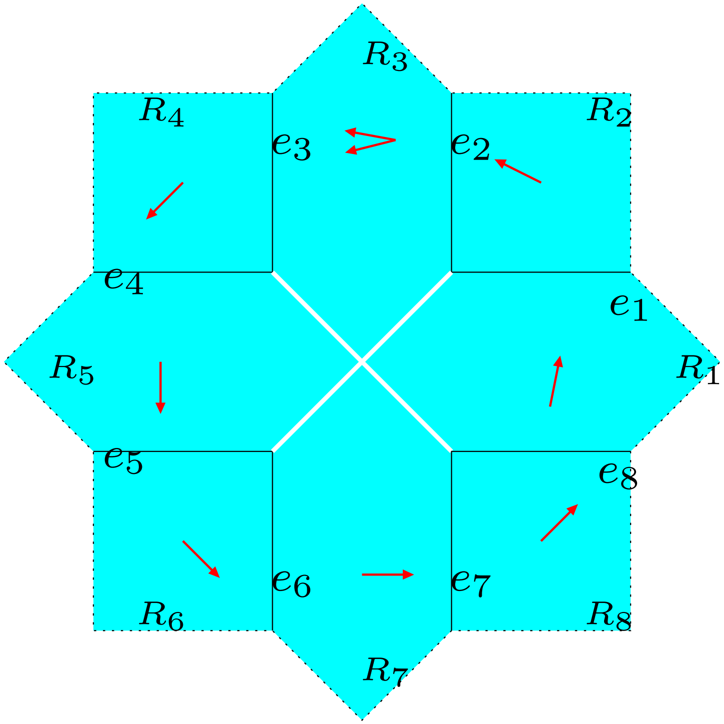
Final Point	*HyTech*	*SPeeDI*	*Reachable*
199	`overflow`	`0.05 sec`	Yes
200	`overflow`	`0.05 sec`	No
201	`overflow`	`0.01 sec`	No
210	`overflow`	`0.05 sec`	No
5	`0.04 sec`	`0.05 sec`	No
20	`0.07 sec`	`0.05 sec`	No
$\frac{200}{9}$	`0.10 sec`	`0.05 sec`	Yes
$\frac{201}{9}$	`overflow`	`0.03 sec`	Yes
$\frac{199}{9}$	`0.07 sec`	`0.04 sec`	Yes

Comparison with HyTech

Simulation of reachability for $x_f = \frac{201}{9}$



K_σ



Composition of TAMFs

TAMFs are closed under composition:
For

$$\mathcal{F}_1(x) = F_1(\{x\} \cap S_1) \cap J_1$$

and

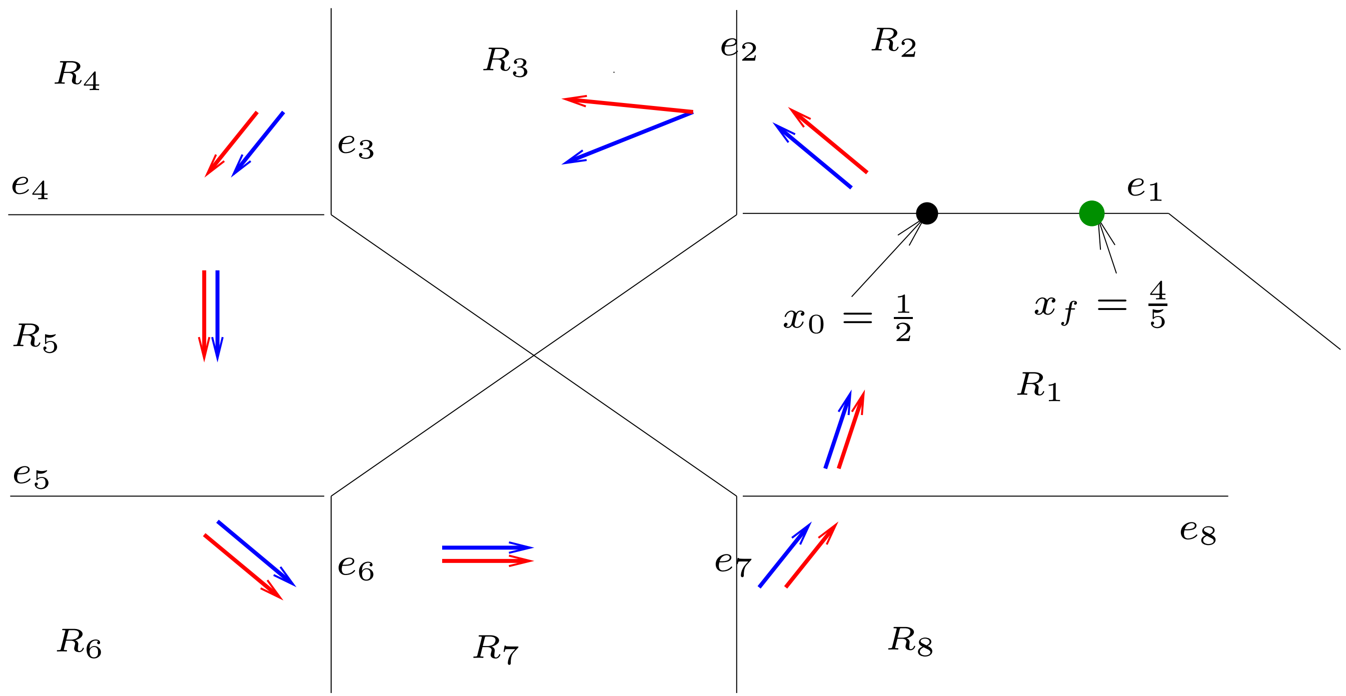
$$\mathcal{F}_2(x) = F_2(\{x\} \cap S_2) \cap J_2$$

we have that

$$\mathcal{F}_2 \circ \mathcal{F}_1(x) = \mathcal{F}_{F',S',J'}(x)$$

with
$F' = F_2 \circ F_1$,
$J' = J_2 \cap F_2(J_1 \cap S_2)$ and
$S' = S_1 \cap F_1^{-1}(J_1 \cap S_2)$

Reachability Algorithm (Example)

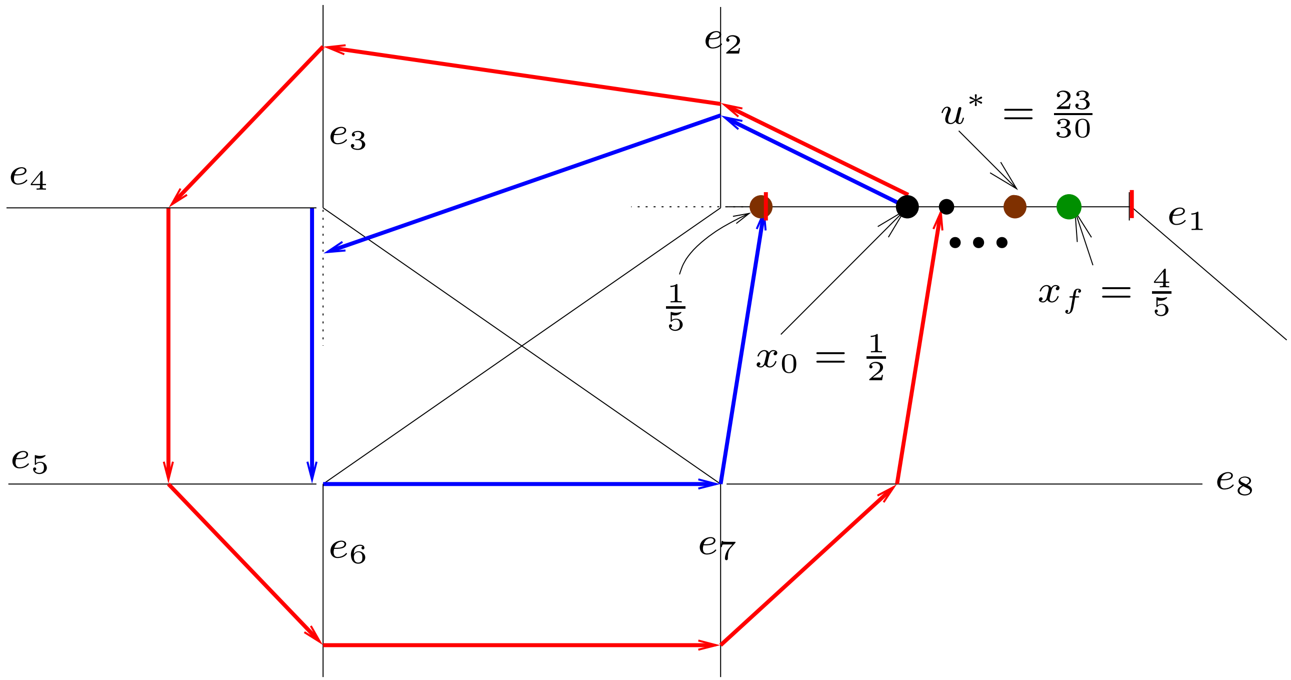


- Type of signature: $\sigma = (e_1 \cdots e_8)^*$
- Successor for the loop $s = e_1 \dots e_8$:

$$\mathsf{Succ}_s(l, u) = [\frac{l}{2} - \frac{1}{20}, \frac{u}{2} + \frac{23}{60}] \cap (\frac{1}{5}, 1)$$
$$\text{if } [l, u] \subseteq (0, 1)$$

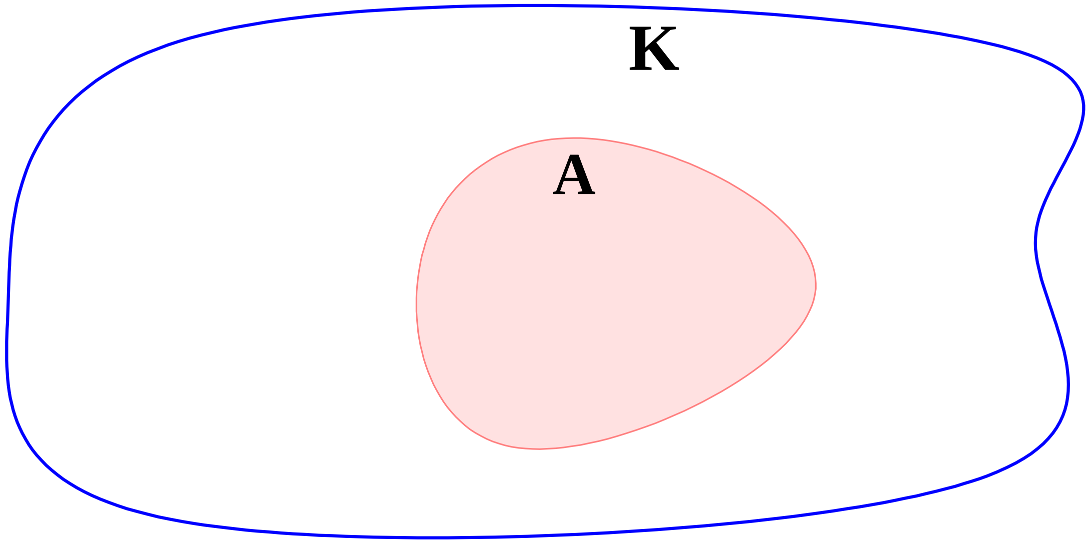
Reachability Algorithm (Example)

- Fixpoint equation: $\mathsf{Succ}_{e_1 \ldots e_8}(I^*) = I^*$
- Solution: $I^* = [l^*, u^*] = [\frac{1}{5}, \frac{23}{30}]$
- Hence: $\mathsf{Succ}_{e_1 \ldots e_8}(x_0) \subseteq [\frac{1}{5}, \frac{23}{30}]$

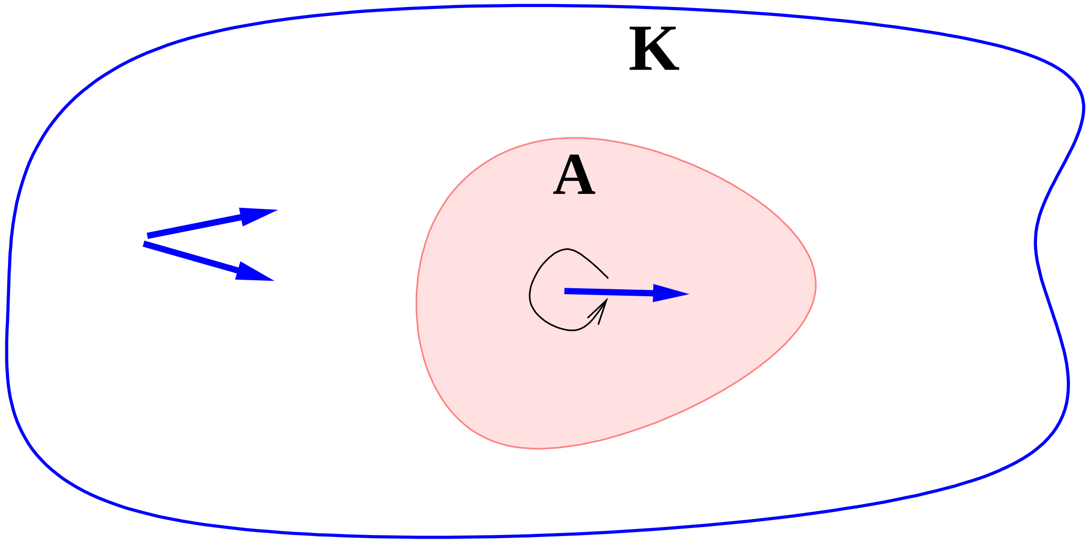


Conclusion: $x_f \notin [\frac{1}{5}, \frac{23}{30}]$).

Viability Kernel



Viability Kernel



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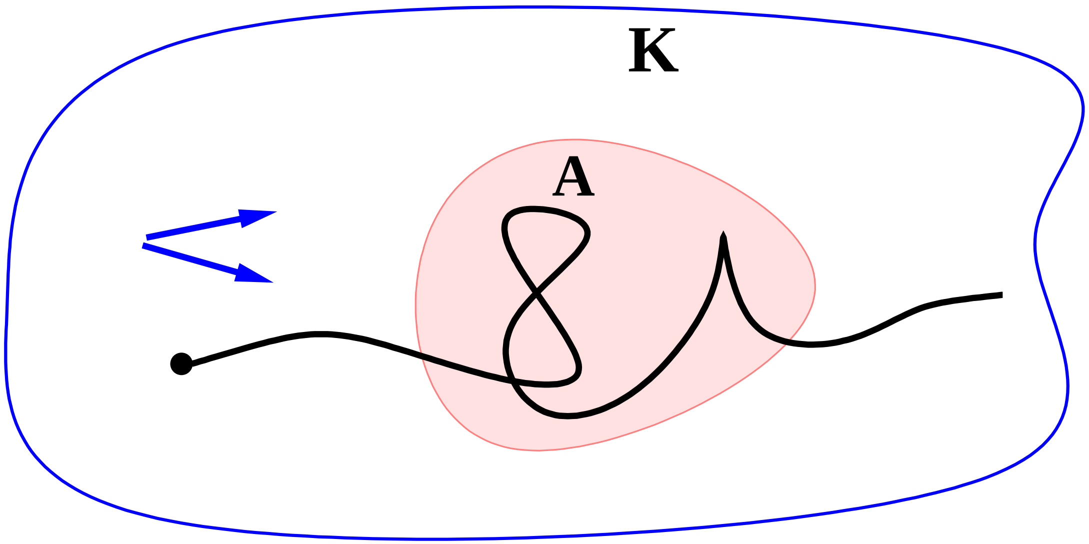
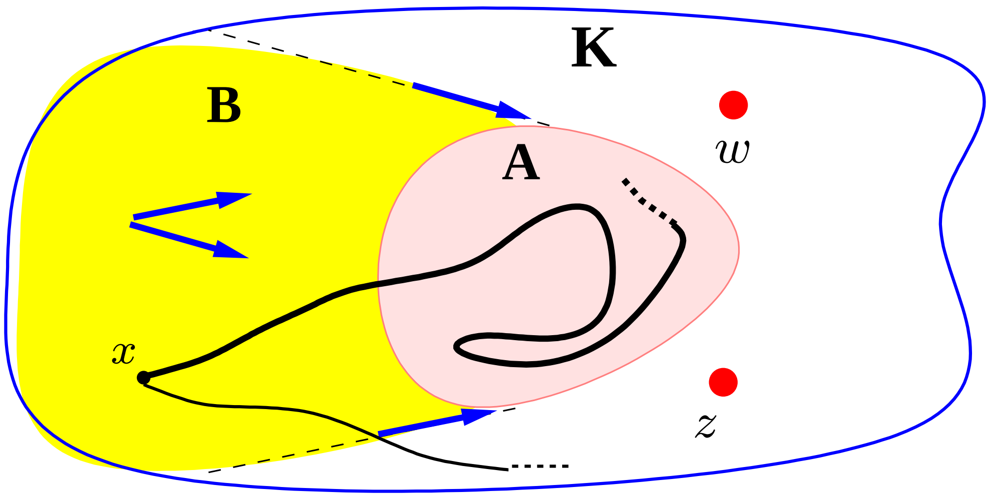


Viability Kernel



$$\mathsf{Viab}(K) = A \cup B$$

- M is a *viability domain* if $\forall \mathbf{x} \in M$, $\exists$ at least one trajectory ξ, starting in $\mathbf{x}$ and remaining in M
- $\mathsf{Viab}(K)$: *Viability kernel* of K is the largest viability domain M contained in K

Viability Kernel for SPDIs

- We can easily compute the Viability Kernel for one cycle, which is a polygon

Viability Kernel for SPDIs

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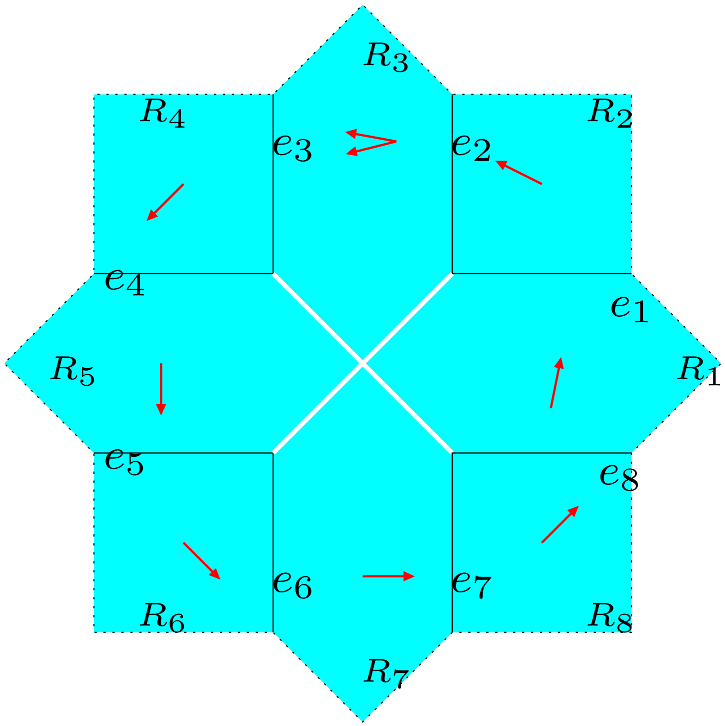

Viability Kernel for SPDIs

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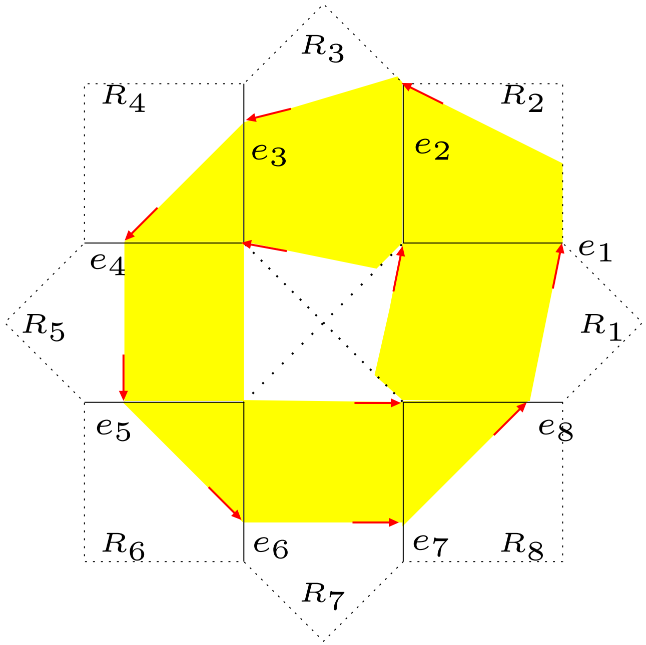
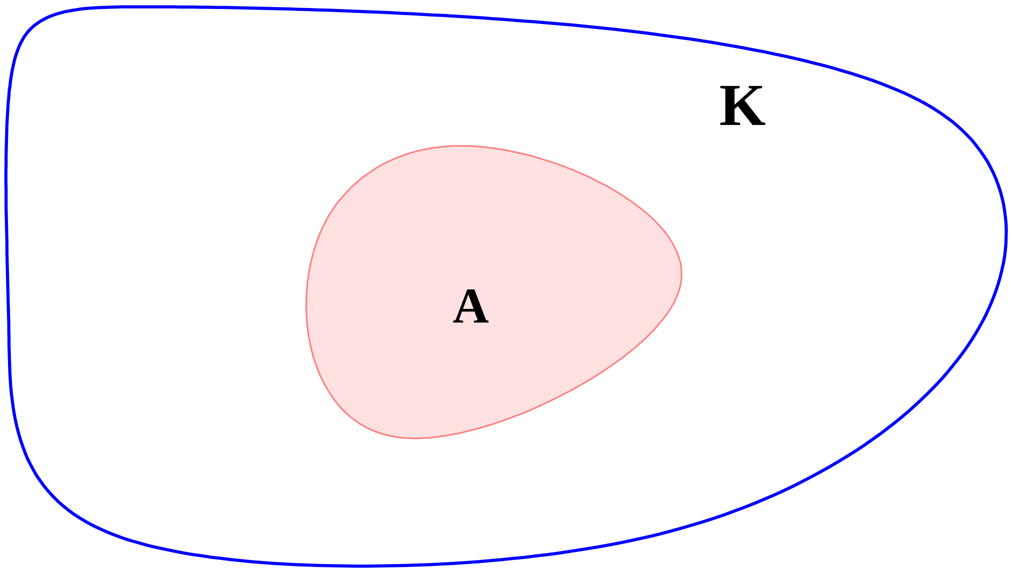
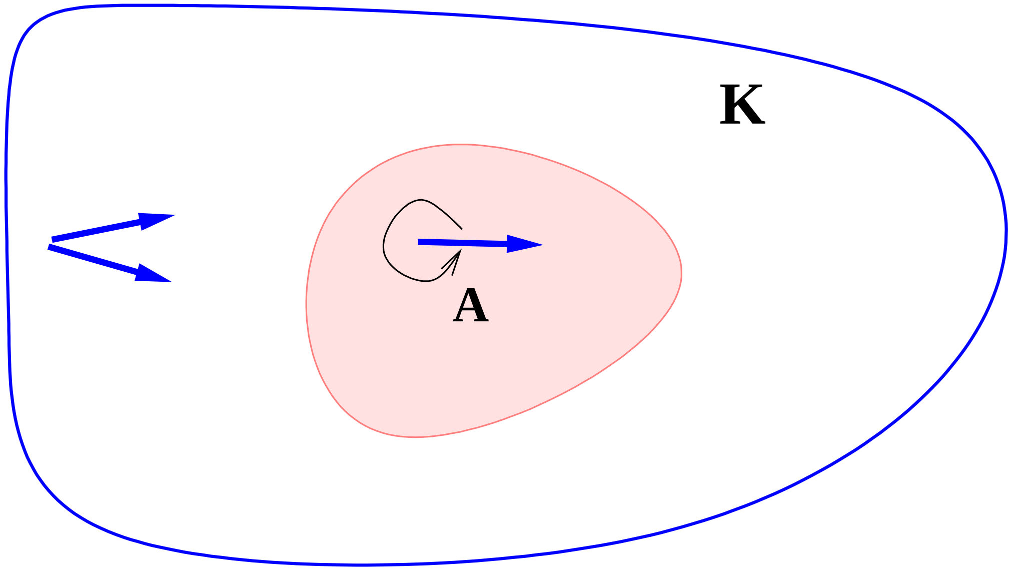


- **Theorem:** $\mathsf{Viab}(K_\sigma) = \overline{\mathsf{Pre}}_\sigma(\mathsf{Dom}(\mathsf{Succ}_\sigma))$

Controllability Kernel



Controllability Kernel



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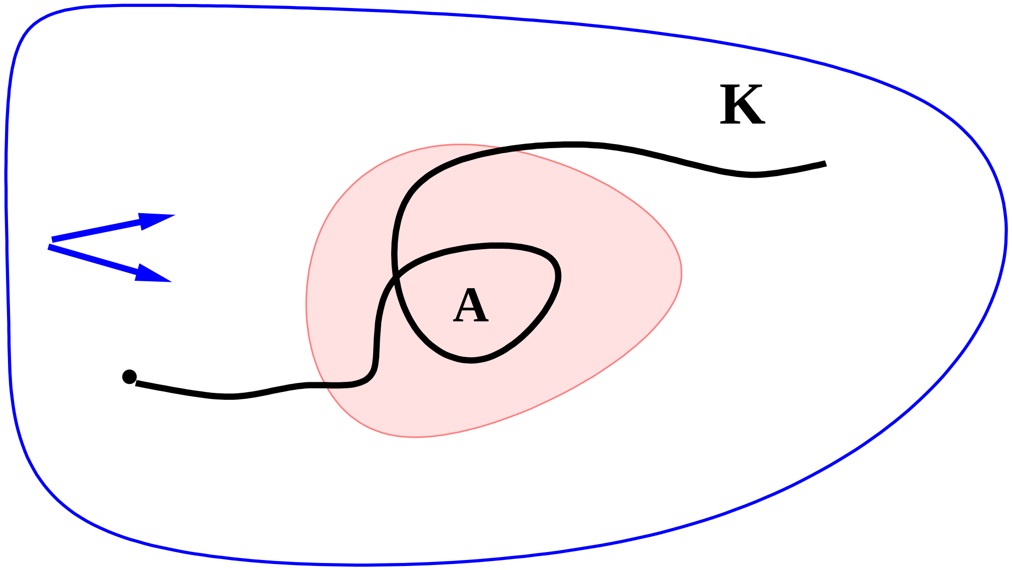

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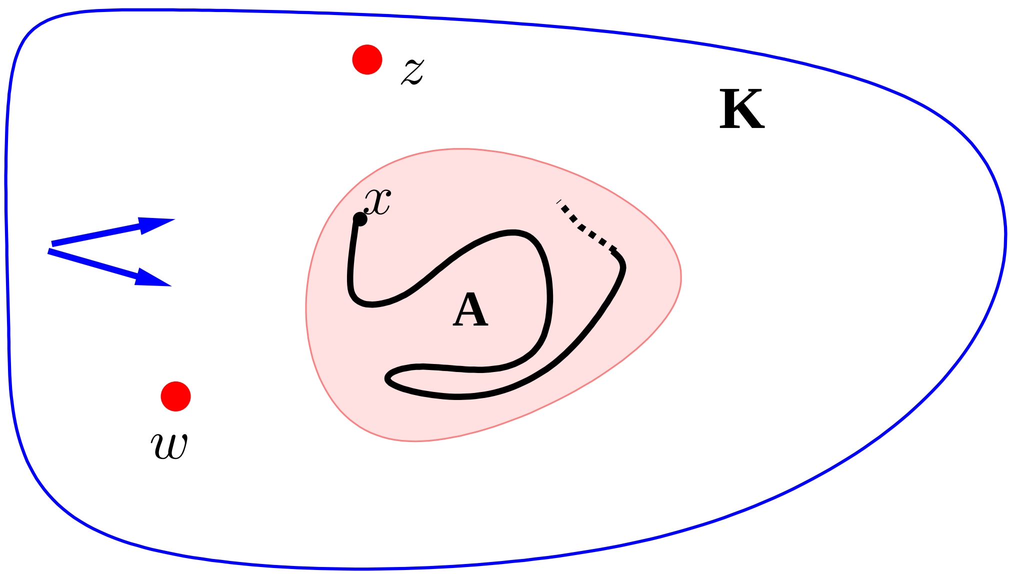
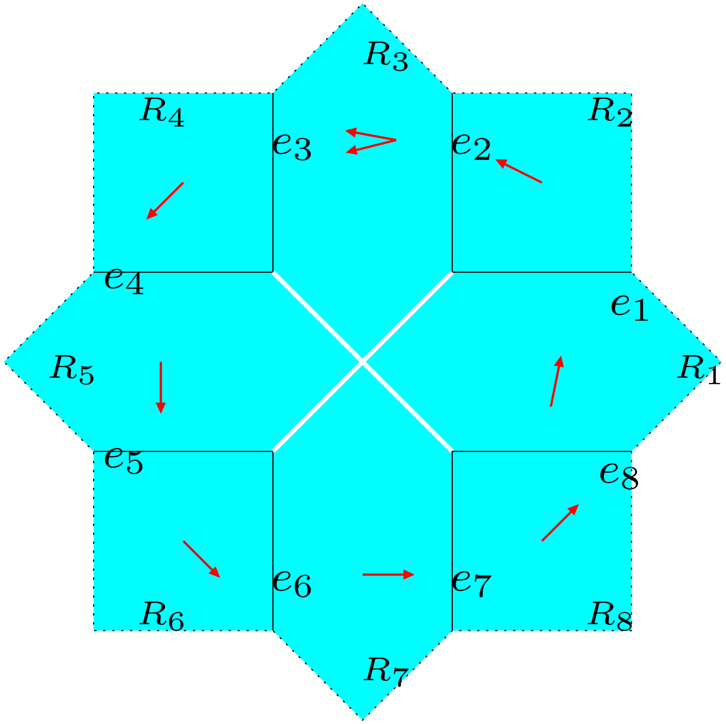


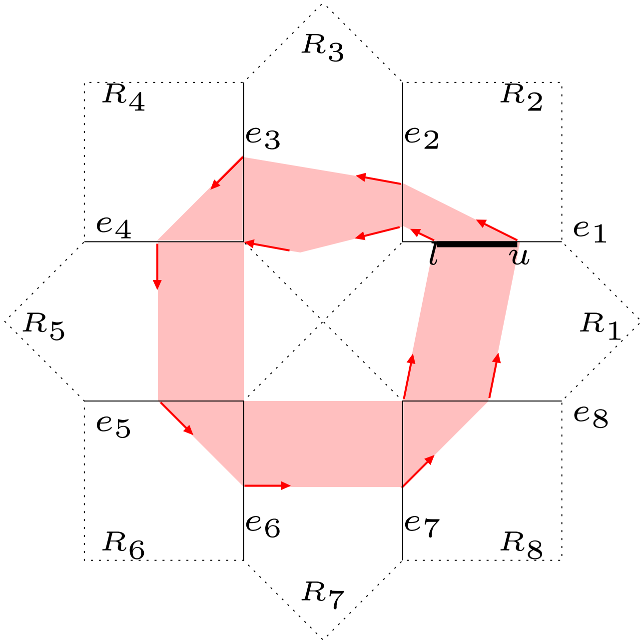
$$\mathsf{Cntr}(K) = A$$

- M is *controllable* if $\forall \mathbf{x}, \mathbf{y} \in M$, $\exists$ a trajectory segment ξ starting in $\mathbf{x}$ that reaches an arbitrarily small neighborhood of $\mathbf{y}$ without leaving M
- *Controllability kernel* of K, denoted $\mathsf{Cntr}(K)$, is the largest controllable subset of K

Controllability Kernel for SPDIs

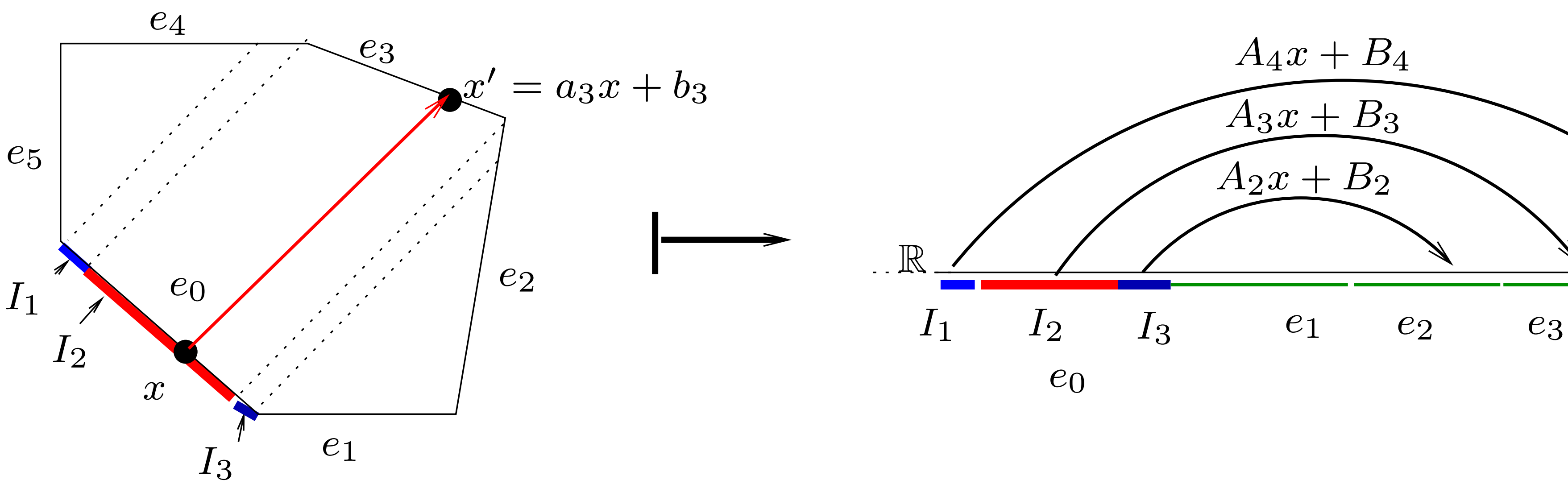


Controllability Kernel for SPDIs

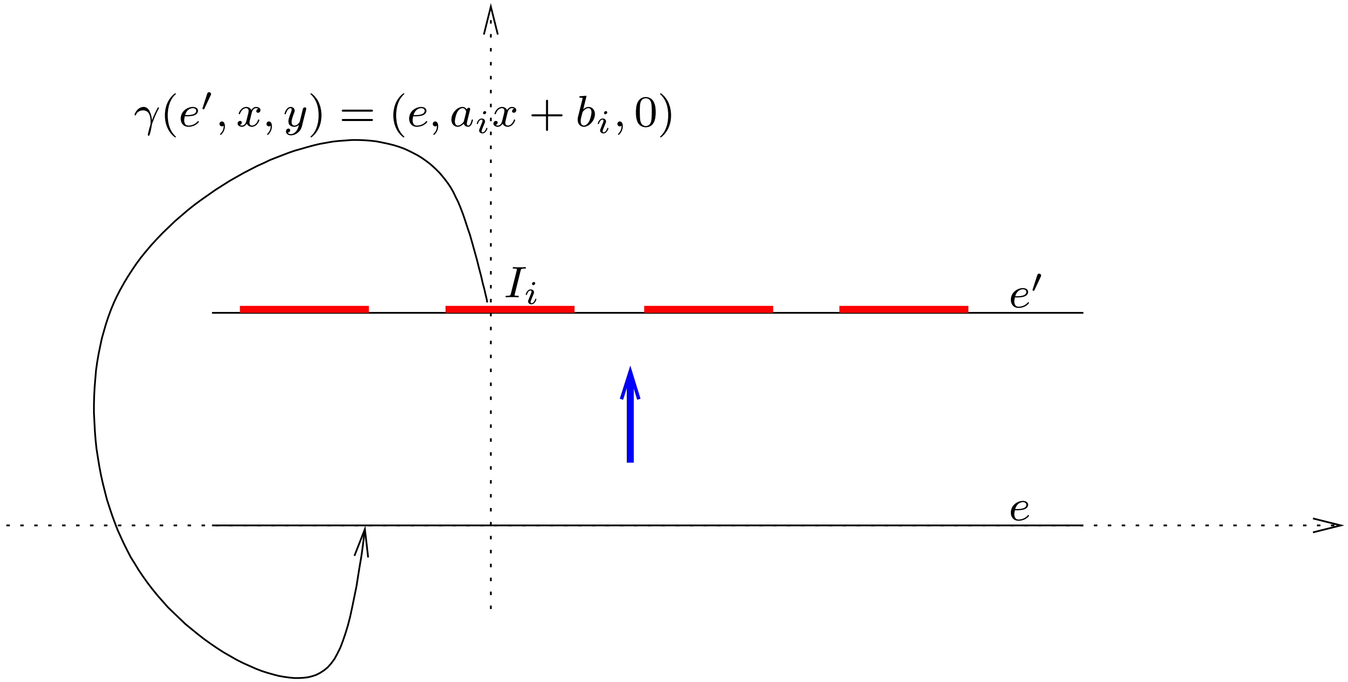


- **Theorem:** $\mathsf{Cntr}(K_\sigma) = (\overline{\mathsf{Succ}}_\sigma \cap \overline{\mathsf{Pre}}_\sigma)(\mathcal{C}_\mathcal{D}(\sigma))$
  (We know how to compute the special interval $\mathcal{C}_\mathcal{D}(\sigma) = [l, u]$)

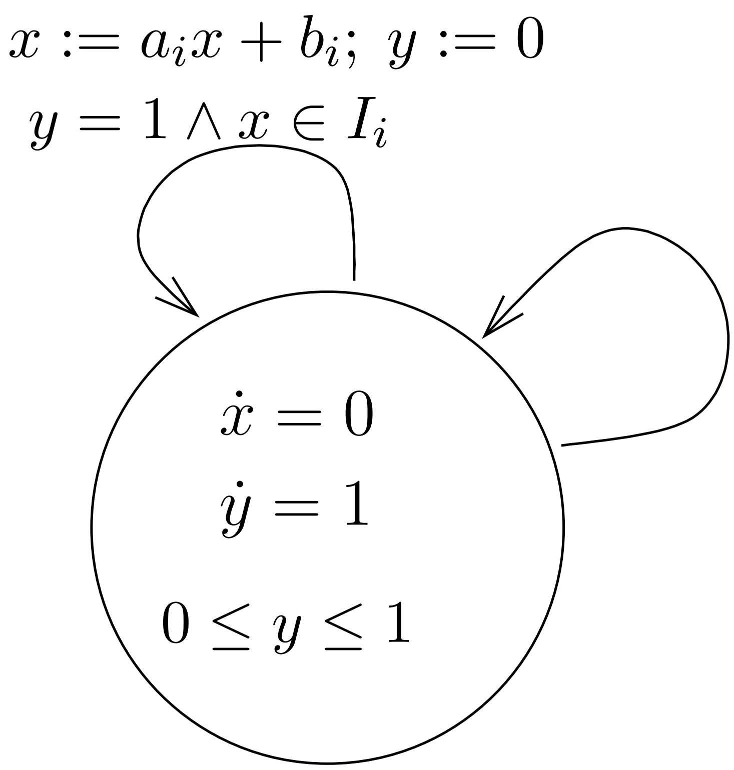
PAM simulate HPCD



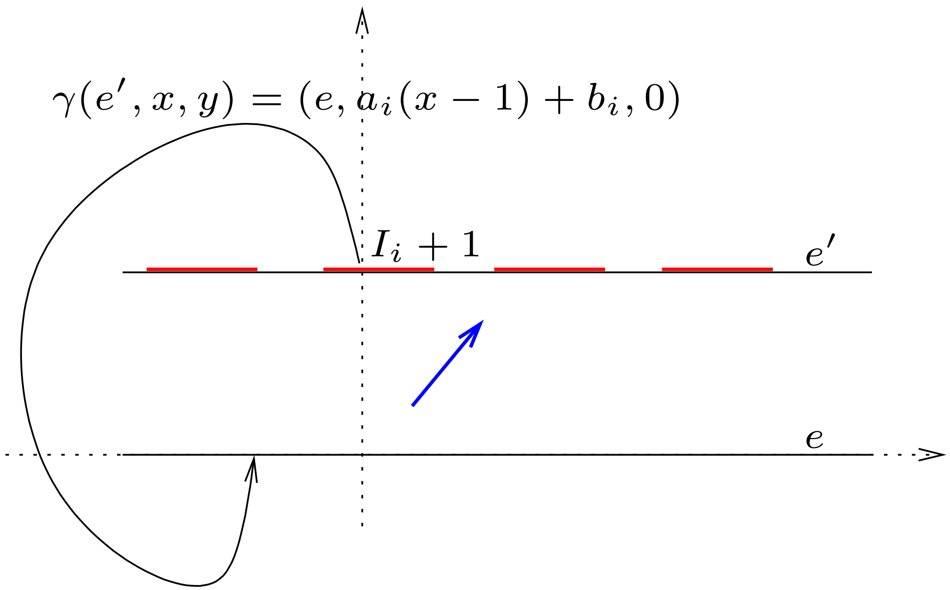
HPCD simulate PAM

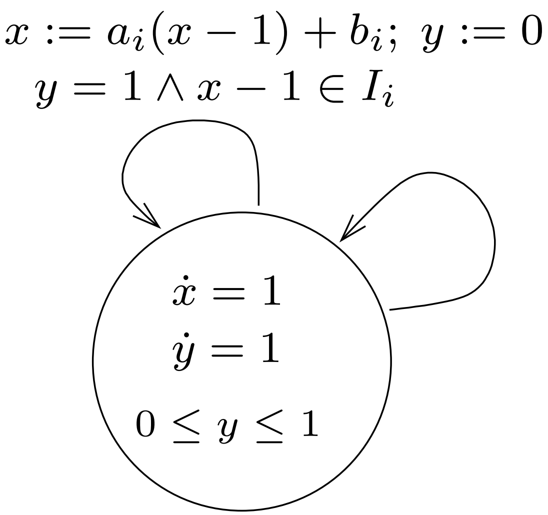


$\mathrm{RA}_{\mathrm{1cl1mc}}$ **equivalent to PAM**

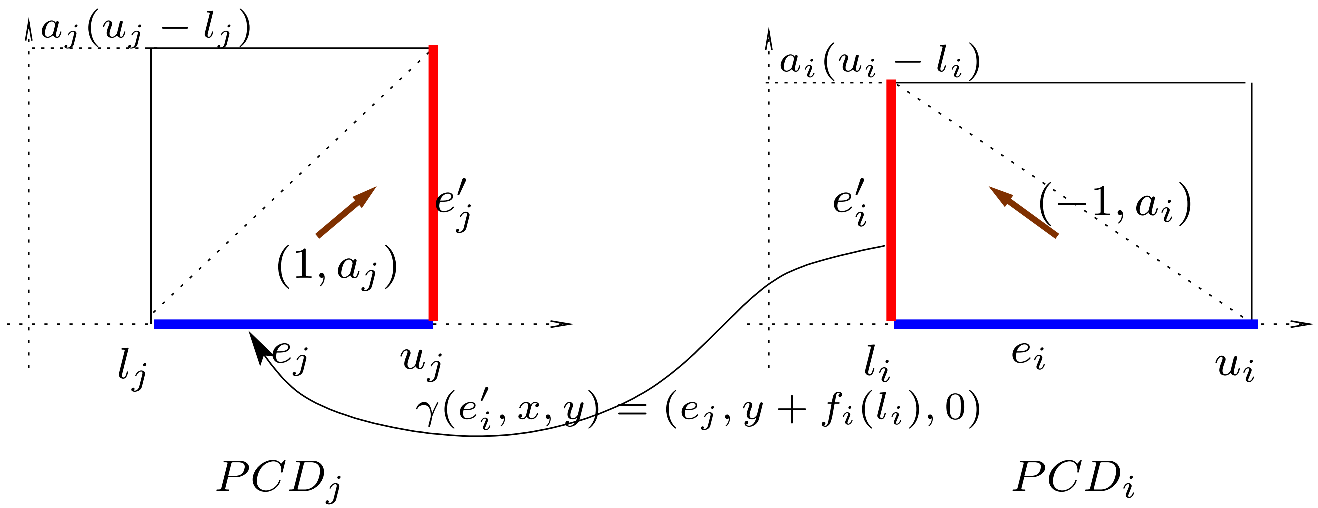


$\mathrm{RA}_{2\mathrm{cl}}$ **equivalent to PAM**

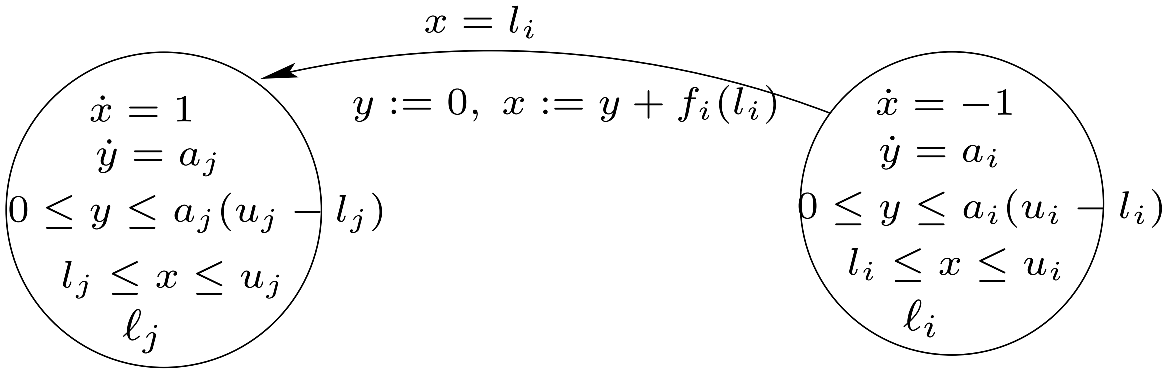




RA_{1sk1sl} **equivalent to PAM**

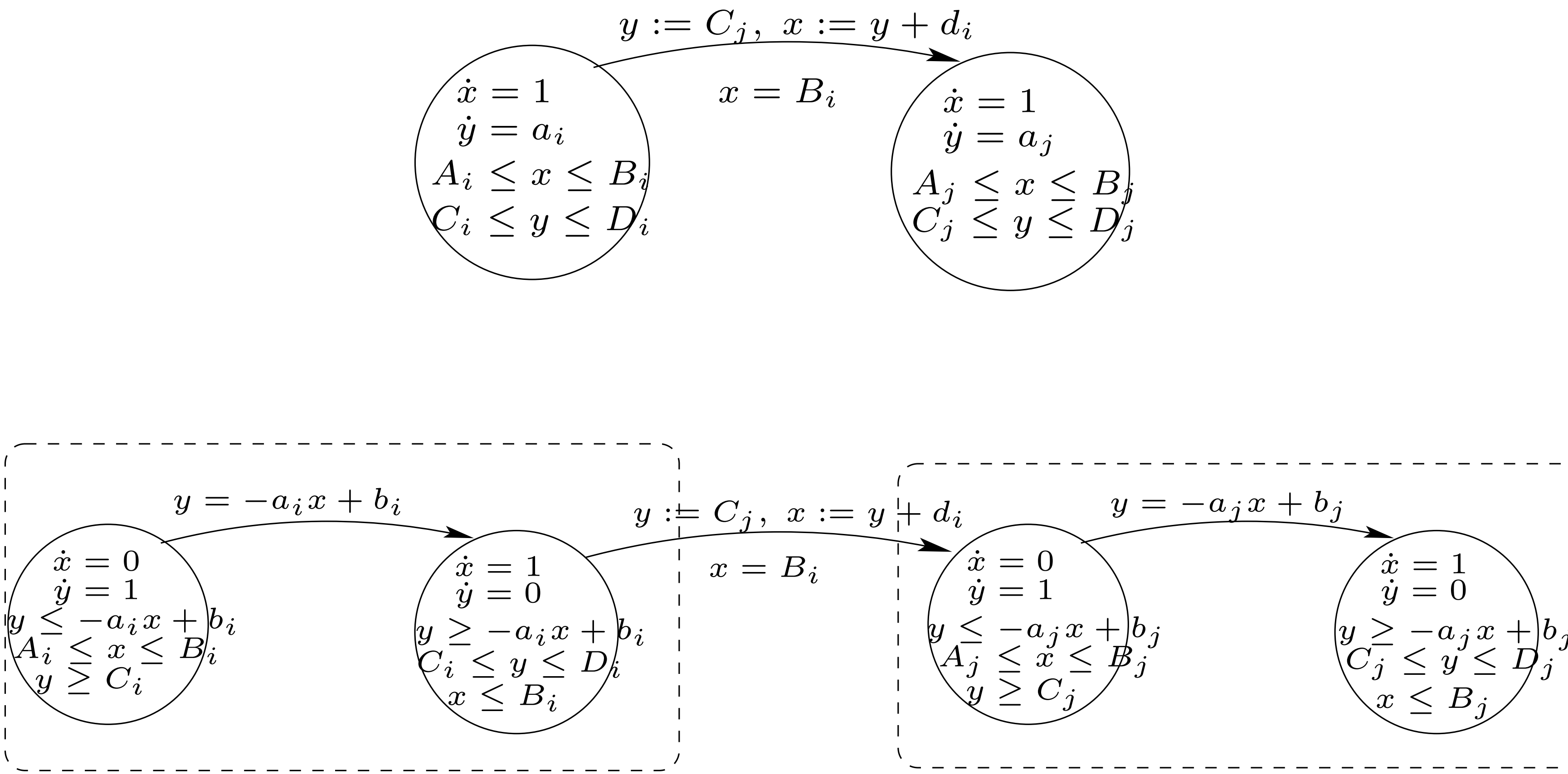


(a)



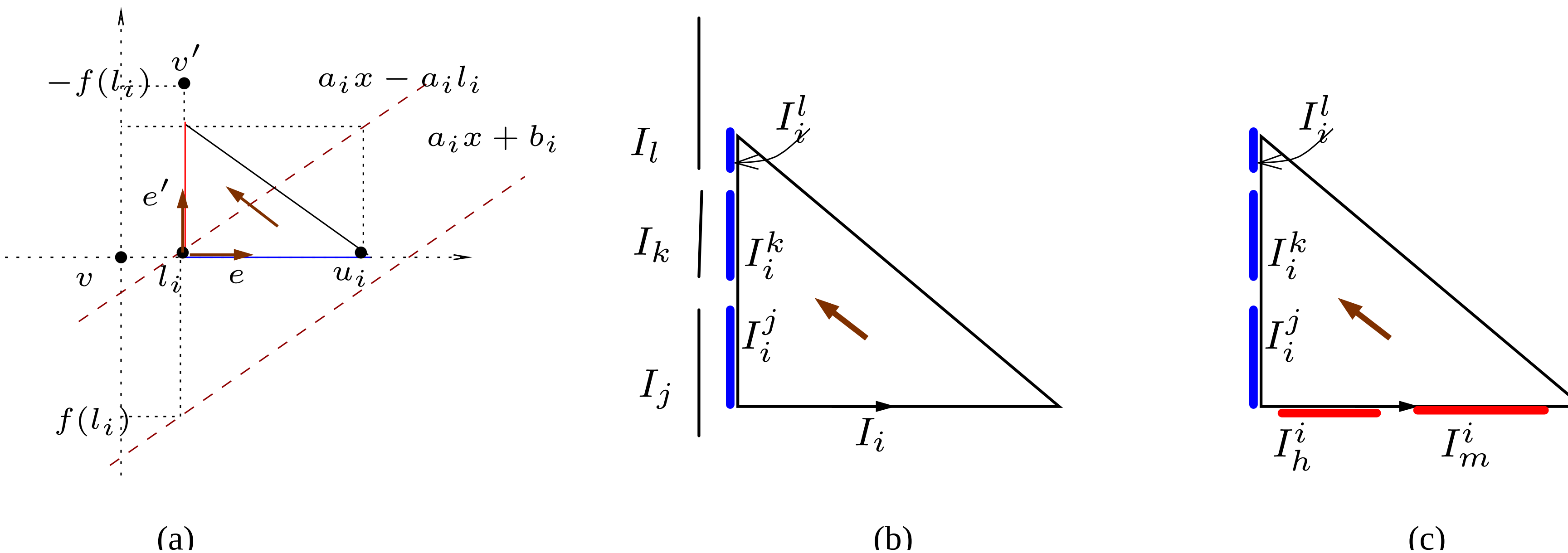
(b)

From $\mathrm{RA}_{1\mathrm{sk}1\mathrm{sl}}$ to $\mathrm{LA}_{\mathrm{st}}$

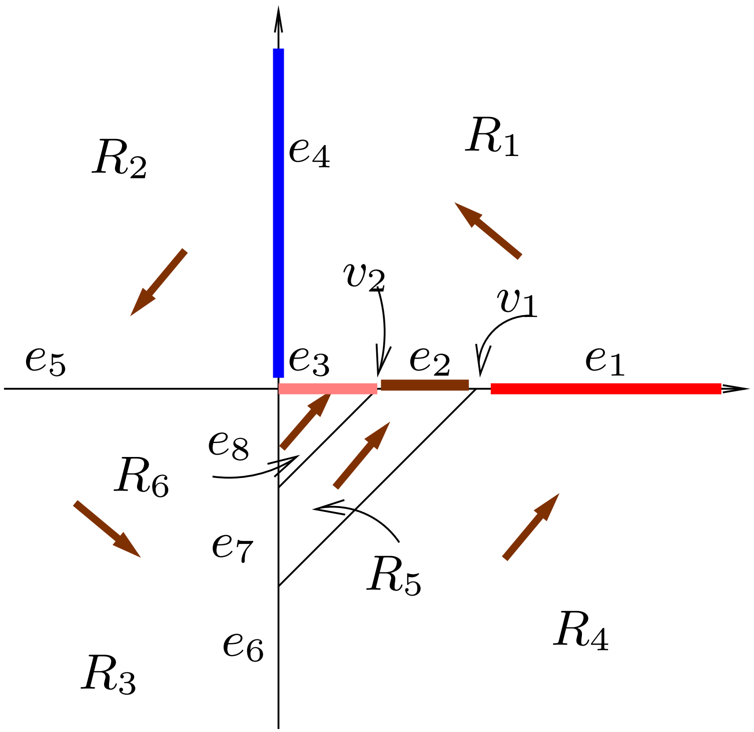


where $b_i = C_i + B_i a_i$ and $b_j = C_j + B_j a_j$

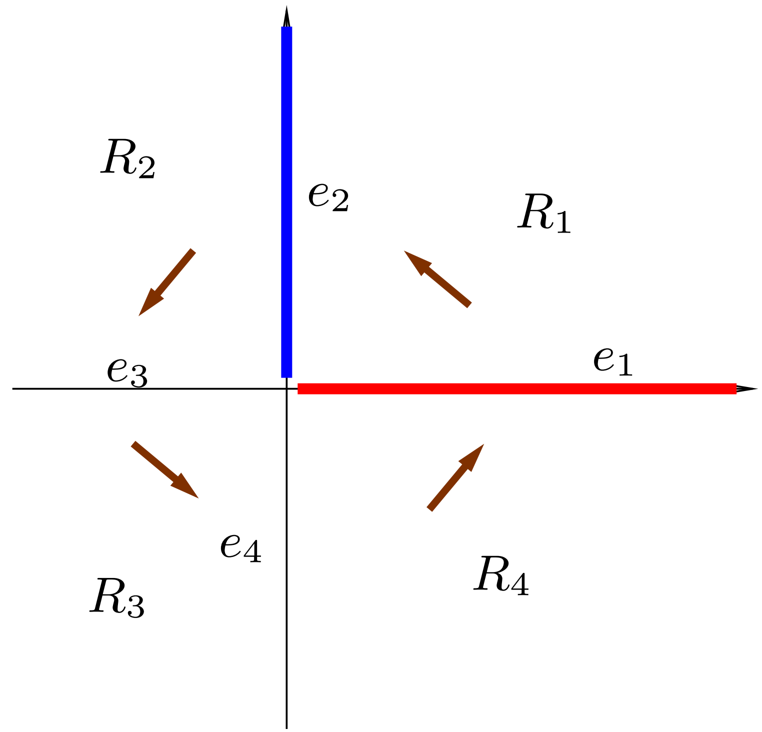
PCD_{2m} **simulate** PAM_{inj}



HPCD_{1c} **simulate TM**



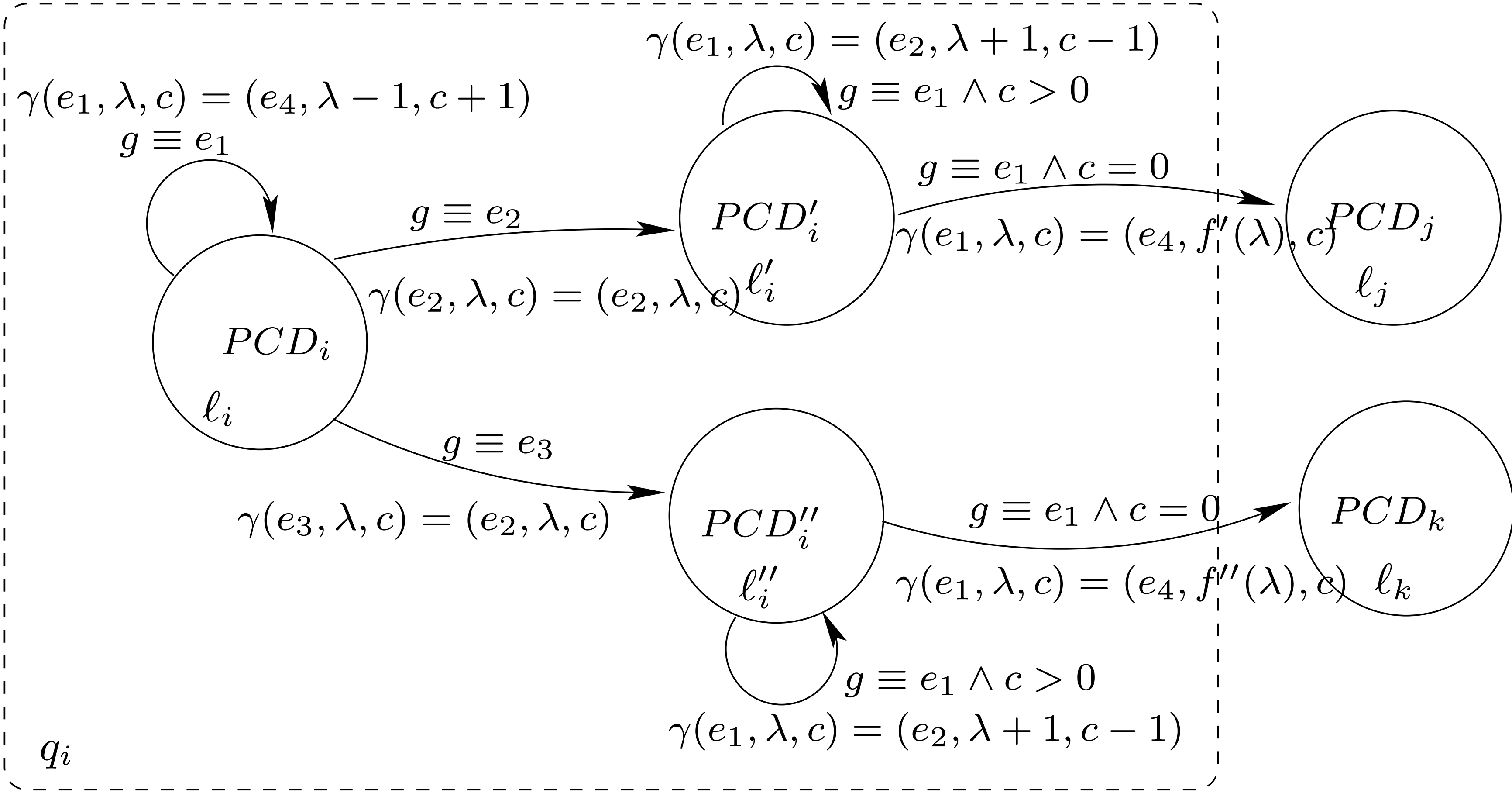
PCD_i



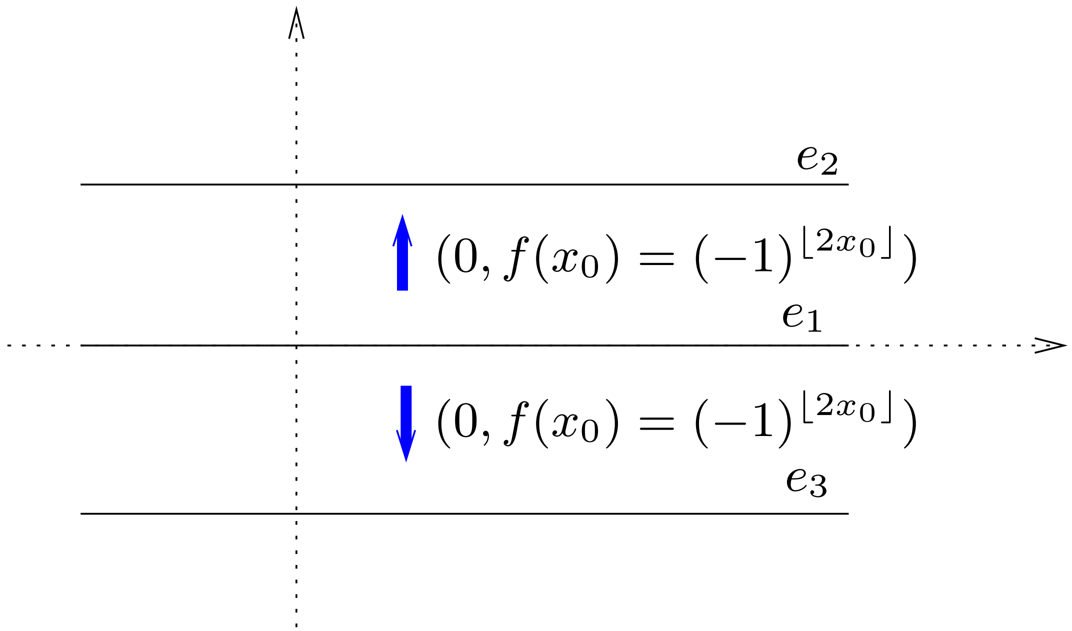
$PCD_i'; PCD_i''$

HPCD_{1c} **simulate TM**

TM-state q_i:



HPCD$_x$ simulate TM



$$f(x) = \begin{cases} 1 & \text{if } \mathrm{frac}_x < \frac{1}{2} \\ -1 & \text{otherwise} \end{cases}$$