A category of cubical sets

Introduction

This note presents a notion of cubical set and the notion of *composition* structure that one can associate to these cubical sets. Any cubical set with a composition structure is fibrant. This universe is closed by dependent product and sum, identity types and data types. Furthermore, it is univalent, and has itself a composition structure.

Cubical sets

Base category

The base category C is the full subcategory of the category of posets having for objects finite power of the poset $0 \leqslant 1$. We write $[1]$ the poset $0 \leqslant 1$. We write $I, J, K, \ldots$ the object of C and $1_I : I \to I$ the identity map of I. If $f : J \to I$ and $g : K \to J$ we write $fg : K \to I$ their composition. If I is an object of C, we have two constant maps $c_0 : I \to [1]$ and $c_1 : I \to [1]$. We write $\pi_1 : I \times [1] \to I$ and $\pi_2 : I \times J \to [1]$ the projection maps and if $f : I \to J$ and $g : I \to [1]$ we write $(f, g) : I \to J \times [1]$ the pairing map. For any object I we define $e_0 = (1_I, c_0) : I \to I \times [1]$ and $e_1 = (1_I, c_1) : I \to I \times [1]$. We may write I^+ instead of $I \times [1]$ and $f^+ : J^+ \to I^+$ the map $f^+(j, b) = (f\ j, b)$. We have the lattice operations $\wedge, \vee : [1]^2 \to [1]$.

Cubical sets

A *cubical set* X is a presheaf on C. It is given by a family of sets $X(I)$ together with restriction maps $X(I) \to X(J),\ u \longmapsto uf$ such that $u1_I = u$ and $(uf)g = u(fg)$ for $f : J \to I$ and $g : K \to J$. (We write uf for what is usually written $X(f)(u)$.)

Sieves

If I is an object of C, a *sieve* L on I is a set of maps $\alpha : J \to I$ of codomain I such that αg is in L whenever α is in L for $\alpha : J \to I$ and $g : K \to J$. If L is a sieve on I and $f : J \to I$ we define the sieve Lf on J to be the set of maps $\beta : K \to J$ such that $f\beta$ is in L.

We define $\Omega(I)$ to be the set of sieves on I. This defines a cubical set (which is the subobject classifier of the topos of presheaves).

Definition 0.1 If L is in $\Omega(I \times [1])$ we define $\forall L$ in $\Omega(I)$ to be the sieve of maps $\alpha : J \to I$ such that α^+ is in L.

If L is a sieve on I and X is a cubical set, we define the set $X(L)$ to be the set of families u_α in $X(J)$ for α in L, such that $u_\alpha g = u_{\alpha g}$ if $g : K \to J$. If u is an element of $X(L)$ and $f : J \to I$, we define uf element of $X(Lf)$ by $uf_\beta = u_{f\beta}$.

Each element $f : I \to [1]$ determines a sieve $[f = 0]$ on I of maps $g : J \to I$ such that $fg = c_0$, and a sieve $[f = 1]$ of maps $g : J \to I$ such that $fg = c_1$. We define the subpresheaf $\mathbb{F}$ of Ω by taking $\mathbb{F}(I)$ to be the set of finite union of sieves of the form $[f = 0] \cap [g = 1]$.

Lemma 0.2 *If L is in $\mathbb{F}(I \times [1])$ then $\forall(L)$ is in $\mathbb{F}(I)$.*

Informal comment: I am not yet sure how to best present the proof of this Lemma, There is a natural notion of face maps in the base category. A face map is a map e_0, e_1 and if f is a face map then so is $f+$. One can then show that a sieve is in $\mathbb{F}(I)$ if and only if it is generated by face maps of codomain I.

Composition structure on a cubical set

If X is a cubical set, we define what is a *composition* structure c_X for X.

It is given by an operation $c_X(I, L, u, a_0)$ producing an element in $X(I)$ and taking as arguments

1. an object I
2. a sieve L in $\mathbb{F}(I)$
3. a family $u_\alpha \in X(J \times [1])$ for $\alpha : J \to I$ in L such that $u_\alpha g = u_{\alpha g^+}$ if $g : K \to J$
4. an element a_0 in $X(I)$ such that $a_0\alpha = u_\alpha e_0$ in $X(J)$ for $\alpha : J \to I$ in L.

The element $a_1 = c_X(I, L, u, a_0)$ should be such that $a_1\alpha = u_\alpha e_1$.

Furthermore, we have the uniformity condition $c_X(I, L, u, a_0)f = c_X(J, Lf, uf, a_0f)$ in $X(J)$ for $f : J \to I$ where $uf_\beta = u_{f\beta}$ for β in Lf.

(Intuitively, the family u and the element a_0 defines an open box, and this operation build the missing lid of an open box in X. We recover the usual Kan composition operation in the special case where L is the boundary of I.)

We also require a similar family of operations where we swap 0 and 1.

Fibrant cubical sets

If X is a cubical set we say that X is *fibrant* if we can "fill any open box of X": we have an operation $\mathsf{fill}(I, L, u, a_0)$ producing an element in $X(I \times [1])$ such that $\mathsf{fill}(I, L, u, a_0)e_0 = a_0$ in $X(I)$ and $\mathsf{fill}(I, L, u, a_0)\alpha^+ = u_\alpha$ in $X(J \times [1])$ for $\alpha : J \to I$ in L.

Proposition 0.3 *If X has a composition structure, then X is fibrant. We have an operation $\mathsf{fill}(I, L, u, a_0)$ such that $\mathsf{fill}(I, L, u, a_0)e_0 = a_0$ and $\mathsf{fill}(I, L, u, a_0)e_1 = \mathsf{comp}(I, L, u, a_0)$ in $X(I)$. This operation is furthermore* uniform, *in the sense that we have $\mathsf{fill}(I, L, u, a_0)f^+ = \mathsf{fill}(J, Lf, uf, a_0f)$ if $f : J \to I$.*

Proof. We define $\mathsf{fill}(I, L, u, a_0)$ to be $\mathsf{comp}(I \times [1], L', u', a'_0)$ where L' is in $\mathbb{F}(I \times [1])$ and u'_β in $X(J \times [1])$ for $\beta : J \to I \times [1]$ in L' and $a'_0 = a_0\pi_1$ in $X(I \times [1])$. We define L' to be the set of maps $\beta : J \to I \times [1]$ such that $\pi_1\beta$ is in L or $\pi_2\beta = c_0$. We define then u'_β by case:

1. if $\beta = (\alpha, \omega)$ with α in L, then we have to define u'_β in $X(J \times [1])$. We have u_α in $X(J \times [1])$ and we take $u'_\beta = u_\alpha(1_J, \delta)$ with $\delta : J \times [1] \to [1]$ is defined by $\delta(j, b) = \omega(j) \wedge b$
2. if $\beta = (g, c_0)$ we define $u'_\beta = a_0g\pi_1$ in $X(J \times [1])$

This definition is coherent since if $\beta = (\alpha, c_0)$ then $u'_\beta = u_\alpha e_0\pi_1 = a_0\alpha\pi_1$.

We have $u'_\beta e_0 = a'_0\beta$ in both cases. If $\beta = (\alpha, \omega)$ then $u'_\beta e_0 = u_\alpha\delta e_0 = u_\alpha e_0 = a_0\alpha = a_0\pi_1\beta = a'_0\beta$. If $\beta = (g, c_0)$ then $u'_\beta e_0 = a_0g\pi_1 e_0 = a_0g = a_0\pi_1\beta = a'_0\beta$.

We can then compute $\mathsf{comp}(I \times [1], L', u', a'_0)e_0 = u'_{e_0}e_0 = a_0$ and, by uniformity

$$\mathsf{comp}(I \times [1], L', u', a'_0)e_1 = \mathsf{comp}(I, L'e_1, u'e_1, a_0) = \mathsf{comp}(I, L, u, a_0)$$

since $L'e_1 = L$ and $u'e_1 = u$.

This operation is uniform. Indeed if $f : J \to I$ we have

$$(a_0f)' = a_0f^+ \qquad (Lf)' = L'f^+ \qquad (uf)' = u'f^+$$

The first equality follows from $f\pi_1 = \pi_1 f^+$. For the second equality, if $\gamma : K \to J \times [1]$ we have γ in $(Lf)'$ if, and only if, $\pi_1\gamma$ is in Lf, which is equivalent to $f\pi_1\gamma = \pi_1 f^+\gamma$ in L i.e. γ in Lf^+, or $\pi_2\gamma = c_0$, which is equivalent to $\pi_2 f^+\gamma = c_0$. Finally, we check that we have $(uf)' = u'f^+$ in $X(K \times [1])$. Given $\gamma = (\alpha, \omega) : K \to J \times [1]$ the element $(uf)'_\gamma$ is defined by case. If α is in Lf then it is $uf_\alpha \delta e_0 = u_{f\alpha}\delta e_0$. In this case, we also have

$$(u'f^+)_\gamma = u'_{f^+\gamma} = u'_{(f\alpha,\omega)} = u_{f\alpha}\delta e_0$$

In the case where $\omega = c_0$ we have $(uf)'_\gamma = a_0 f\alpha\pi_1$ which is equal to $(u'f^+)_\gamma = u'_{(f\alpha,\omega)} = a_0 f\alpha\pi_1$. $\square$

Universe of cubical sets

We fix a Grothendieck universe $\mathcal{U}$.

If I is an object of C, we define $U(I)$ to be the collection of all presheaves $(\mathsf{C}/I)^{op} \to \mathcal{U}$. An element A of $U(I)$ is given by a family of $\mathcal{U}$-sets A_f, for $f : J \to I$, together with restriction maps $A_f \to A_{fg},\ u \longmapsto ug$ for $g : K \to J$, such that $u1_J = u$ and $(ug)h = u(gh)$ if $h : L \to K$.

If A is an element of $U(I)$ and $f : J \to I$ we can consider the element Af of $U(J)$ defined by $Af_g = A_{fg}$. We have $A1_I = A$ and $(Af)g = A(fg)$ if $g : K \to J$.

If A and B are in $U(I)$ we define a map $\sigma : A \to B$ to be a family of set-theoretic maps $\sigma_f : A_f \to B_f$ for $f : J \to I$ satisfying the naturality condition $(\sigma_f u)g = \sigma_{fg}(ug)$ if $g : K \to J$ and u is in A_f. We may write simply $\sigma : A_f \to B_f$ and the naturality condition becomes $(\sigma u)g = \sigma(ug)$.

Composition structure

If A is an element of $U(I)$ we define what is a *composition* structure c_A for A. It is given by an operation $c_A(f, L, u, a_0)$ producing an element in A_{fe_1} and taking as arguments

1. a map $f : J \times [1] \to I$
2. an element L in $\mathbb{F}(J)$
3. a family $u_\alpha \in A_{f\alpha^+}$ such that $u_\alpha g^+ = u_{\alpha g}$ if $\alpha : K \to J$ in L and $g : H \to K$
4. an element a_0 in A_{fe_0} such that $a_0\alpha = u_\alpha e_0$ in $A_{fe_0\alpha}$.

The element $a_1 = c_A(f, L, u, a_0)$ should satisfy $a_1\alpha = u_\alpha e_1$.

Furthermore, we have the uniformity condition $c_A(f, L, u, a_0)g = c_A(fg^+, Lg, ug, a_0 g)$ in $A_{fe_1 g}$ if $g : K \to J$.

We also require a similar family of operations where we swap 0 and 1.

We write $CS(A)$ the set of composition structure on A.

If c_A is an element of $CS(A)$ and $f : J \to I$ we can define a composition structure $c_A f$ on $CS(Af)$ by taking $c_A f(g, L, u, a_0) = c_A(fg, L, u, a_0)$.

Lemma 0.4 *If c_A is in $CS(A)$ then $c_A f$ is in $CS(Af)$, and we have $c_A 1_I = c_A$ and $(c_A f)g = c_A(fg)$ if $g : K \to J$.*

Fibrant objects

If A is an element in $U(I)$ we say that A is *fibrant* if we can fill any open box of A: we have an operation $\mathsf{fill}(f, L, u, a_0)$ producing an element in A_f such that $\mathsf{fill}(f, L, u, a_0)e_0 = a_0$ and $\mathsf{fill}(f, L, u, a_0)\alpha^+ = u_\alpha$.

Proposition 0.5 *If A in $U(I)$ has a composition structure, then A is fibrant. More precisely, we have an operation $\mathsf{fill}(c_A, f, L, u, a_0)$ producing an element in A_f such that $\mathsf{fill}(c_A, f, L, u, a_0)e_0 = a_0$ and $\mathsf{fill}(c_A, f, L, u, a_0)e_1 = c_A(f, L, u, a_0)$. This operation is furthermore* uniform, *in the sense that we have $\mathsf{fill}(c_A, f, L, u, a_0)g^+ = \mathsf{fill}(c_A, fg^+, Lg, ug, a_0 g)$ if $g : K \to J$.*

Glueing operation

If M is in $\mathbb{F}(I)$ we define $U(M)$ to be the collection of families T of sets T_α, for α in M, such that $u1_J = u$ if u is in T_α and ug is in $T_{\alpha g}$ if u is in T_α and $g : K \to J$. If T is in $U(M)$ and $f : J \to I$ we define Tf by $Tf_\alpha = T_{f\alpha}$ if α is in Mf.

For M in $\mathbb{F}(I)$, the *glueing operation* takes as argument A in $U(I)$, and T in $U(M)$, and a family σ of maps $\sigma_\alpha : T_\alpha \to A_\alpha$ for α in M. This family has to be uniform: $(\sigma_\alpha t)g = \sigma_{\alpha g}(tg)$ if $g : K \to J$. If $f : J \to I$ we define σf by $\sigma f_\alpha = \sigma_{f\alpha}$ for α in Mf. The result of this operation $\mathsf{glue}(A, T, \sigma)$ is then an element in $U(I)$ such that $\mathsf{glue}(A, T, \sigma)f = Tf$ if f is in M.

For $f : J \to I$ we define the set $\mathsf{glue}(A, T, \sigma)_f$ by (decidable) case

1. if f is in M we take $\mathsf{glue}(A, T, \sigma)_f = T_f$
2. otherwise $\mathsf{glue}(A, T, \sigma)_f$ is the set of element (u, t) where u is in A_f and t is a family t_β in $T_{f\beta}$ for $\beta : K \to J$ in Mf and $\sigma_{f\beta} t_\beta = u\beta$ and $t_\beta h = t_{\beta h}$ for $h : L \to K$.

We then define, for $g : K \to J$, the element $(u, t)g$ by case. If fg is in M, we take t_g. Otherwise we take (ug, tg) with $tg_\gamma = t_{g\gamma}$ for γ in Mfg.

This defines an element $\mathsf{glue}(A, T, \sigma)$ in $U(I)$.

Lemma 0.6 *The map $\sigma : T \to A$ can be extended to a map $\delta : B \to A$*

Proof. Given $f : J \to I$ we have to define a set-theoretic map $\delta : B_f \to A_f$. If f is in M we have $B_f = T_f$ and we take $\delta = \sigma$. If f is not in M then v in B_f is a pair (a, t) with a in A_f and we take $\delta(a, t) = a$. We have to verify that $(\delta v)g = \delta(vg)$ for $g : K \to J$. If f is in M then v is in T_f and $(\delta v)g = (\sigma v)g = \sigma(vg) = \delta(vg)$. If f is not in M there are two cases. If fg is in M then $(\delta v)g = ag$ and $vg = t_g$ with $\sigma t_g = \delta t_g = ag$. If fg is not in M then $vg = (ag, tg)$ and $\delta(vg) = ag$. $\square$

Equivalence structure

An *equivalence structure* on σ is given by two operations $q^1_\sigma(f, L, u, b)$ in A_f and $q^2_\sigma(f, L, u, b)$ in $B_{f\pi_1}$ and taking as arguments

1. $f : J \to I$
2. L in $\mathbb{F}(J)$
3. a family of elements u_α in $A_{f\alpha}$ for $\alpha : K \to J$ in L such that $u_\alpha g = u_{\alpha g}$ in $A_{f\alpha g}$ if $g : H \to K$
4. an element b in B_f such that $b\alpha = \sigma u_\alpha$ in $B_{f\alpha}$ if α is in L

We should have

$$q^1_\sigma(f, L, u, b)\alpha = u_\alpha \qquad q^2_\sigma(f, L, u, b)e_0 = \sigma\ q^1_\sigma(f, L, u, b) \qquad q^2_\sigma(f, L, u, b)e_1 = b$$

Furthermore we have the uniformity conditions

$$q^1_\sigma(f, L, u, b)g = q^1_\sigma(fg, Lg, ug, bg) \qquad q^2_\sigma(f, L, u, b)g^+ = q^2_\sigma(fg, Lg, ug, bg)$$

if $g : K \to J$.

If $\sigma : A \to B$ and $f : J \to I$ we can define $\sigma f : Af \to Bf$ by taking $(\sigma f)_g = \sigma_{fg}$ and if q^1_σ, q^2_σ is an equivalence structure on σ we define $q^1_\sigma f, q^2_\sigma f$ equivalence structure on σf by taking $q^i_\sigma f(g, L, u, b) = q^i_\sigma(fg, L, u, b)$ if $g : K \to J$.

Lemma 0.7 *Given A, T in $U(I)$, and c_A (resp. c_T) a comsposition structure on A (resp. T). Let σ be a map $T \to A$. Assume furthermore given*

1. *a map $f : J \times [1] \to I$*

2. *an element L in $\mathbb{F}(J)$*

3. *a family $v_\alpha \in T_{f\alpha^+}$ such that $v_\alpha g^+ = v_{\alpha g}$ if $\alpha : K \to J$ in L and $g : H \to K$*

4. *an element t_0 in T_{fe_0} such that $t_0\alpha = v_\alpha e_0$ in $T_{fe_0\alpha}$.*

We can build $u = \mathsf{pres}(c_A, c_T, L, v, t_0)$ in A_f such that

$$ue_0 = c_A(f, L, \sigma v, \sigma t_0) \qquad ue_1 = \sigma c_T(f, L, v, t_0) \qquad u\alpha^+ = \sigma v_\alpha e_1 \pi_1$$

Furthermore, $\mathsf{pres}(c_A, c_T, L, v, t_0)g^+ = \mathsf{pres}(c_A g, c_T g, Lg, vg, t_0 g)$ if $g : K \to J$.

If L is in $\mathbb{F}(I)$, we can generalize the notion of composition structure for an element of $U(L)$ and the notion of equivalence structure for a map between two elements of $U(L)$. We can now refine the operation of glueing in the following way.

Theorem 0.8 *Given L in $\mathbb{F}(I)$, A in $U(I)$, and T in $U(L)$, and a map σ between T and A, we can build a composition structure $\mathsf{glue}(c_A, c_T, q_\sigma)$ on $\mathsf{glue}(A, T, \sigma)$ given a composition structure c_A on A and a composition structure c_T on T and an equivalence structure q_σ on σ in such a way that we have $\mathsf{glue}(c_A, c_T, q_\sigma)\alpha = c_T\alpha$ if α is in L*

If L is in $\mathbb{F}(I)$ and T, A are in $U(L)$ and σ is a map $T \to A$ then for each $f : J \to I$ in L we can consider Tf, Af in $U(J)$ and the map $\sigma f : Tf \to Af$.

Proof of the main Theorem

The goal of this section is to prove Theorem 0.8. We write $B = \mathsf{glue}(A, T, \sigma)$ and want to define a composition structure c_B on B.

Using Lemma 0.6, the map $\sigma : T \to A$ extends to a map $\delta : B \to A$.

We give $f : J \times [1] \to I$ and M in $\mathbb{F}(J)$ and v_α in $B_{f\alpha^+}$ for α in M and b_0 in B_{fe_0} such that $b_0\alpha = v_\alpha e_0$ for α in M. We want to compute $b_1 = c_B(f, M, v, b_0)$ in B_{fe_1} such that $b_1\alpha = v_\alpha e_1$ for α in M.

We define $a_0 = \delta b_0$ and $u_\alpha = \delta v_\alpha$. Since δ is a map $B \to A$ we have $a_0\alpha = u_\alpha e_0$. We can then form $a_1' = c_A(f, M, u, a_0)$ which satisfies $a_1'\alpha = u_\alpha e_1$ for α in M.

We can consider three sieves on J. One is the given sieve M. From the sieve Lf on $J \times [1]$ we can derive the sieve Lfe_1 in $\mathbb{F}(J)$. We can also define the sieve $N = \forall(Lf)$ of maps $\beta : K \to J$ such that β^+ is in Lf. Notice that N is a subsieve of Lfe_1: if $f\beta^+$ is in L then so is $f\beta^+e_1 = fe_1\beta$. By Lemma 0.2 we know that N is in $\mathbb{F}(J)$.

The universe of types

If I is an object of C we let $U_F(I)$ be the set of element (A, c_A) where A is in $U(I)$ and c_A is in $CS(A)$. If $f : J \to I$ we define $(A, c_A)f = (Af, c_Af)$ which is an element of $U_F(J)$ by Lemma 0.4. In this way we define a new cubical set U_F.

Lemma 0.9 *Given E in $U(I \times [1])$ and a composition structure c_E on E we can define $A = Ee_0$, $B = Ee_1$ in $U(I)$ and c_Ee_0 is in $CS(A)$ and c_Ee_1 is in $CS(B)$. We can also define a map $\sigma : A \to B$ by $\sigma a = \mathsf{comp}(f, \emptyset, \emptyset, a)$ in B_f for a in A_f and $f : J \to I$ and σ has an equivalence structure.*

Notice that if E is of the form $A\pi_1$ with A in $U(I)$ then $B = Ee_1 = A$ and this map $\sigma : A \to A$ does not need to be the identity map.

We can use Theorem 0.8 and Lemma 0.9 to prove the following result.

Theorem 0.10 *The cubical set U_F has a composition operation.*

References

[1] V. Voevodsky. Notes on type systems. github.com/vladimirias/old_notes_on_type_systems, 2009-2012.