Types as Kan Simplicial Sets

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Introduction

The usual justification of the Kan simplicial set model of type theory uses classical logic in an essential way. We present a constructive version of this model. We restrict ourselves to the dimension $\leqslant 1$ but this should hopefully extends to all dimensions.

1 Syntax

We have the judgements $\Gamma \vdash \quad \Gamma \vdash A \quad \Gamma \vdash a : A$ and $\sigma : \Delta \to \Gamma$

The main rules are

$$\frac{\sigma : \Delta \to \Gamma \quad \Gamma \vdash A}{\Delta \vdash A\sigma} \qquad \frac{\sigma : \Delta \to \Gamma \quad \Gamma \vdash a : A}{\Delta \vdash a\sigma : A\sigma} \qquad \frac{\Gamma \vdash A \quad \Gamma.A \vdash B}{\Gamma \vdash \Pi\ A\ B}$$

$$\frac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda b : \Pi\ A\ B} \qquad \frac{\Gamma \vdash c : \Pi\ A\ B \quad \Gamma \vdash a : A}{\Gamma \vdash c\ a : B[a]}$$

where $[a] : \Gamma \to \Gamma.A$ is the substitution $[a] = (1, a)$. The main conversion rule is $(\lambda b)\ a = b[a]$.

2 Constructive Kan Fibrations

A simplicial set is a pair $X, X^{(1)}$ with a map $\eta_0 : X \to X^{(1)}$ and two maps $d_0 \quad d_1 : X^{(1)} \to X$ such that $d_i\ (\eta_0\ x) = x$

If α satisfies $d_i\ \alpha = a_i$ we write $\alpha : a_0 \to a_1$ and we think of α as a *line* joining the *points* a_0 and a_1

We define a triangle θ to be a triple of lines $\alpha_0 : a_1 \to a_2,\ \alpha_1 : a_0 \to a_2,\ \alpha_2 : a_0 \to a_1$. We define $d_i\ \theta = \alpha_i$. We define a *square* Δ as a pair of triangles θ, θ' such that $d_1\ \theta = d_1\ \theta'$. We see this square as a line between two lines $d_2\ \theta$ and $d_0\ \theta'$. The square θ', θ can also be seen as a line between the two lines $d_2\ \theta'$ and $d_0\ \theta$. If we think of θ as a triangle $a_0a_1a_2$ and θ' as a triangle $a_0b_1a_2$ then $\Delta = \theta, \theta'$ can be seen as a line between a_0a_1 and b_1a_2 and θ', θ as a line between a_0b_1 and a_1a_2. We write $\Delta : a_0a_1 \to b_1a_2$.

The *interval* is the simplicial set I with $0 \quad 1 : I$ and $I^{(1)}$ having one non degenerate element $\alpha : 0 \to 1$. A line of X is then a point of X^I and a square is a line of X^I.

For having the *Kan property* we require 4 compositions operation (and not only 3)

1. $\mathsf{comp}_0\ \alpha\ \beta : a_1 \to a_2$ for $\alpha : a_0 \to a_1,\ \beta : a_0 \to a_2$
2. $\mathsf{comp}_1\ \alpha\ \beta : a_0 \to a_2,\ \ \mathsf{comp}'_1\ \alpha\ \beta : a_0 \to a_2$ for $\alpha : a_0 \to a_1,\ \beta : a_1 \to a_2$
3. $\mathsf{comp}_2\ \alpha\ \beta : a_0 \to a_1$ for $\alpha : a_0 \to a_2,\ \beta : a_1 \to a_2$

The required equations are

$$\mathsf{comp}_0\ (\eta_0\ a)\ \beta = \beta \quad \mathsf{comp}_1\ (\eta_0\ a)\ \beta = \beta$$

$$\mathsf{comp}'_1\ \alpha\ (\eta_0\ b) = \alpha \quad \mathsf{comp}_2\ \alpha\ (\eta_0\ b) = \alpha$$

If B is a simplical set, we define a Kan fibration over B as a dependent family $F(b)$ for b in B together with a set $F(\alpha)$ for each $\alpha : b_0 \to b_1$ with two maps $d_i : F(\alpha) \to F(b_i)$. If ω is in $F(\alpha)$ we think of ω as a line between $d_0\ \omega$ in $F(b_0)$ and $d_1\ \omega$ in $F(b_1)$. Furthermore we have for each u in $F(b)$ a line $\eta_0\ u$ in $F(\eta_0\ b)$ such that $d_i\ (\eta_0\ u) = u$.

The total space F of the fibration has the points the pairs (b, u) with u in $F(b)$. A line between b_0, u_0 and b_1, u_1 is a pair α, ω with $\alpha : b_0 \to b_1$ and $\omega : u_0 \to u_1$. The projection $p(b, u) = b$ is then a simplical map.

We require also a map $F(\alpha)^+ : F(b_0) \to F(b_1)$ and a line $F(\alpha) \uparrow u : u \to F(\alpha)^+\ u$. Furthermore $F(\eta_0\ b)^+ u = u$ and $F(\eta_0\ b) \uparrow u = \eta_0\ u$.

But this is not enough. We require also a map $F(\Delta)^+ : F(\alpha_0) \to F(\alpha_1)$ and a square $F(\Delta) \uparrow \omega : \omega \to F(\Delta)^+\ \omega$ for any $\omega : F(\alpha_0)$. Write $\Delta = a_0a_1a_2, a_0b_1a_2$; we have the following degeneracy conditions. If $a_0a_1 = \eta_0\ a_0$ and $b_1a_2 = \eta_0\ b_1$ and $a_1a_2 = a_0b_2 = a_0b_1$ and ω is degenerate then $F(\Delta) \uparrow \omega$ is degenerate. If $a_0b_1 = \eta_0\ a_0$, $a_1a_2 = \eta_0\ a_1$ and $a_0a_1 = a_0a_2 = b_1a_2$ then $F(\Delta)^+\omega = \omega$ and $F(\Delta) \uparrow \omega$ is degenerate.

We require symmetric conditions for the maps $F(\alpha)^- : F(b_1) \to F(b_0)$ and lines $F(\alpha) \downarrow v : F(\alpha)^-\ v \to v$.

This can be expressed more concisely as follows, using A. Joyal's notion of left fibration. We consider the pull-back $B^I \times^0_B F$ of the maps $F \to B$ and $B^{d_0} : B^I \to B$. We have a map $\langle p, d_0\rangle : F^I \to B^I \times^0_B F$ and we require that this map has a section s. Furthermore there are two constant maps $c_0 : F \to F^I$ and $c_1 : F \to B^I \times^0_B F$ and we ask that $c_0 = s \circ c_1$.

Constructively, the conditions on the maps $F(\alpha)^+$, $F(\alpha)^-$ are more complex than in the classical case. On the other hand, the remaining filling conditions are quite close to the classical case. If we have a triangle $b_0b_1b_2$ in B, we require to have 4 compositions operations, which satisfies similar degeneracy equations than in the non relative case. For instance if we have u_i in $F(b_i)$ and two lines $\omega_1 = u_0u_1$ and $\omega_2 = u_0u_2$ then we have $\mathsf{comp}_0\ \omega_1\ \omega_2$ lines between u_1 and u_2. Furthermore, if $b_0b_1 = \eta_0\ b_0$ *and* $u_0u_1 = \eta_0\ u_0$ then $\mathsf{comp}_0\ \omega_1\ \omega_2 = \omega_2$.

3 Model

Each context Γ is interpreted by a Kan simplical set, also written Γ. Each type $\Gamma \vdash A$ is interpreted by a Kan fibration over Γ: if $\sigma : \Gamma$ then $A\sigma$ is a set and we have for $\alpha : \sigma_0 \to \sigma_1$ a set of lines $A\alpha$ between points in $A\sigma_0$ and in $A\sigma_1$. Each element $\Gamma \vdash a : A$ is interpreted by a section of this fibration: if $\sigma : \Gamma$ we have $a\sigma$ point in $A\sigma$ and if $\alpha : \sigma_0 \to \sigma_1$ then $a\alpha : A\alpha$ is a line between the points $a\sigma_0$ in $A\sigma_0$ and $a\sigma_1$ in $A\sigma_1$.

We describe the interpretation of function types.

If $\Gamma \vdash A$ and $\Gamma.A \vdash B$ and $\sigma : \Gamma$ then $(\Pi\ A\ B)\sigma$ is the set of functions f with a function $f^{(1)} = \eta_0\ f$ such that

1. if $u : A\sigma$ then $f\ u : B(\sigma, u)$
2. if $\gamma : u_0 \to u_1$ in $A(\eta_0\ \sigma)$ then $f^{(1)}\ \gamma : f\ u_0 \to f\ u_1$
3. we have $f^{(1)}\ (\eta_0\ u) = \eta_0\ (f\ u)$

If $\alpha : \sigma_0 \to \sigma_1$ then $(\Pi\ A\ B)\alpha$ is the set of triple f_0, f_1, λ with $f_i : (\Pi\ A\ B)\sigma_i$ such that if $\omega : u_0 \to u_1$ is in $A\alpha$ then $\lambda\ \omega$ is a path $f_0\ u_0 \to f_1\ u_1$ in $B(\alpha, \omega)$. We define $d_i\ (f_0, f_1, \lambda) = f_i$. We define also $\eta_0\ f = (f, f, f^{(1)})$.

The definition of $g = (\Pi\ A\ B)\alpha^+\ f$ seems to be forced

$$(\Pi\ A\ B)\alpha^+\ f\ v = B(\alpha, A\alpha \downarrow v)^+(f\ (A\alpha^-\ v))$$

The fact that this function $g = (\Pi\ A\ B)\alpha^+\ f$ is continuous, that is that we can define a corresponding function $\eta_0\ g$ is not obvious and that is why we had to introduce further conditions on the definition of constructive Kan fibration.

For this we introduce a degenerate square Δ connecting $\eta_0\ \sigma_0$ and $\eta_0\ \sigma_1$. Given $\omega_1 : v \to v'$ in $A(\eta_0\ \sigma_1)$ we consider the line $A\Delta^-\omega_1 : A\alpha^- v \to A\alpha^- v'$ and we define

$$\eta_0\ g\ \omega_1 = B(\Delta, A\Delta \downarrow \omega_1)^+(\eta_0\ f\ (A\Delta^-\omega_1))$$

Given $\omega : u \to v$ in $A(\alpha)$ we define $\delta = \mathsf{comp}_2\ \omega\ (A\alpha \downarrow v)$ and

$$(\Pi\ A\ B)\alpha \uparrow f\ \omega = \mathsf{comp}_1\ (\eta_0\ f\ \delta)\ (B(\alpha, A\alpha \downarrow v) \uparrow (f\ (A\alpha^- v)))$$

If now Δ is any square in the simplicial set Γ we have to define $(\Pi\ A\ B)\Delta^+$ and $(\Pi\ A\ B)\Delta \uparrow$.

$$(\Pi\ A\ B)\Delta^+\ \lambda\ \omega = B(\Delta, A\Delta \downarrow \omega)^+(\lambda\ (A\Delta^-\ \omega))$$

We see that the definition of $(\Pi\ A\ B)\Delta^+$ is similar to the definition of $(\Pi\ A\ B)\alpha^+$. In the same way the definition of $(\Pi\ A\ B)\Delta \uparrow$ is similar to the definition of $(\Pi\ A\ B)\alpha \uparrow$. For this, we use the fact that compositions of lines can be extended to composition of squares, with corresponding degeneracy equations.

Finally we explain how to define the composition operations on functions. Given a triangle $\sigma_0\sigma_1\sigma_2$ given by three lines

$$\alpha_1 : \sigma_0 \to \sigma_1 \qquad \alpha_2 : \sigma_0 \to \sigma_2 \quad \alpha : \sigma_1 \to \sigma_2$$

in Γ and two lines $\lambda_1 : f_0 \to f_1$ and $\lambda_2 : f_0 \to f_2$ with f_i in $(\Pi\ A\ B)\sigma_i$ we define

$$\mathsf{comp}_0\ \lambda_1\ \lambda_2 : f_1 \to f_2$$

We consider $\omega : u_1 \to u_2$ in $A\alpha$ with u_i in $A\sigma_i$ and we define $\delta : A\alpha_1^- u_1 \to u_2$ by

$$\delta = \mathsf{comp}_1'\ (A\alpha_1 \downarrow u_1)\ \omega$$

We take then

$$\mathsf{comp}_0\ \lambda_1\ \lambda_2\ \omega = \mathsf{comp}_0\ (\lambda_1\ (A\alpha_1 \downarrow u_1))\ (\lambda_2\ \delta)$$

Notice that if $\alpha_1 = \eta_0\ \sigma_0$ and $\lambda_1 = \eta_0\ f_1$ then we have $\delta = \omega$ and

$$\mathsf{comp}_0\ \lambda_1\ \lambda_2\ \omega = \mathsf{comp}_0\ (\lambda_1\ (A\alpha_1 \downarrow u_1))\ (\lambda_2\ \delta) = \lambda_2\ \omega$$

which is compatible with the degeneracy condition $\mathsf{comp}_0\ (\eta_0\ f_1)\ \lambda_2 = \lambda_2$.

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