Types as Kan Simplicial Sets

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Introduction

The usual justification of the Kan simplicial set model of type theory uses classical logic in an essential way. We present a constructive version of this model. We explain then how to give a purely symbolic account, which can be implemented as an extension of type theory.

1 Gandy's interpretation

The model we shall present generalizes Gandy's interpretation of extensional type theory in intensional type theory [4]. It can also be seen as a generalization of Russell's interpretation of Boolean algebra in the theory of implications [8].

However we shall adopt a non standard way to state that a relation is symmetric and transitive. Gandy's interpretation contains in particular a proof that if R is reflexive on X and S is an equivalence relation on Y then the relation $T(f_0, f_1)$ on Y^X

$$R(x_0, x_1) \to S(f_0(x_0), f_1(x_1))$$

is an equivalence relation on elements f satisfying

$$R(x_0, x_1) \to S(f(x_0), f(x_1))$$

With the usual definition of transitivity, there is no canonical proof of the fact that T is transitive. For instance If $T(f, g)$ and $T(g, h)$ for showing that we have $T(f, h)$ we assume $R(a, c)$ and we try to prove $S(f(a), h(c))$. We can then use *either* $S(f(a), g(a))$ and $S(g(a), h(c))$ *or* $S(f(a), g(c))$ and $S(g(c), h(c))$.

Instead, if we express symmetry and transitivity as

$$R(x_0, x_1) \wedge R(x_0, y_0) \wedge R(x_1, y_1) \to R(y_0, y_1)$$

and

$$R(y_0, y_1) \wedge R(x_0, y_0) \wedge R(x_1, y_1) \to R(x_0, x_1)$$

then we get a canonical proof. Indeed if we have $T(f_0, f_1) \wedge T(f_0, g_0) \wedge T(f_1, g_1)$ and we take $R(x_0, x_1)$ we have

$$S(f_0(x_0), f_1(x_1)) \wedge S(f_0(x_0), g_0(x_0)) \wedge S(f_1(x_1), g_1(x_1))$$

and we get $S(g_0(x_0), g_1(x_1))$. Hence we have $T(g_0, g_1)$ as required.

2 Constructive Kan Fibrations

We work in the category of simplicial sets.This category is the presheaf category on the category Δ, whose objects are $[n] = \{0, 1, \dots, n\}$ and maps monotone map. We write Δ_n the functor $\Delta_n([m]) = Hom([m], [n])$. An object can be symbolicaly represented by describing its non degenerate simplex. For instance the interval $I = \Delta_1$ has two points, and one line joining these two points. To give a n-simplex in X is to give a point in X^{Δ_n}. We write $\partial\Delta_n$ the boundary of Δ_n. For instance the boundary of I has two points.

Simplical sets form a topos and in particular a model of dependent types. Any context Γ is interpreted by a simplical set Γ and any dependent type $\Gamma \vdash A$ as a map $\Gamma.A \to \Gamma$. If $\sigma \in \Gamma$ we write $A\sigma$ the fiber of this map above σ.

The first condition we ask is to give the following transfer property

$$\prod_{\alpha:\Gamma^I} A\alpha(0) \to A\alpha(1)$$

We write $A\alpha^+$ the map $A\alpha(0) \to A\alpha(1)$. We ask furthermore that this map satisfies $A\alpha^+ u = u$ whenever α is a constant map $\alpha = \eta_0\ \sigma$.

We ask also the corresponding maps $A\alpha^-$

$$\prod_{\alpha:\Gamma^I} A\alpha(1) \to A\alpha(0)$$

and this map should satisy $A\alpha^- u = u$ whenever α is a constant map $\alpha = \eta_0\ \sigma$.

These conditions are natural if we want to consider a line $\alpha \in \Gamma^I$ as a proof of equality of $\alpha(0)$ and $\alpha(1)$. If we view $A\sigma$ as a proposition dependent on $\sigma \in \Gamma$ then the equality of $\alpha(0)$ and $\alpha(1)$ should imply the logical equivalence of $A\alpha(0)$ and $A\alpha(1)$, which via the identification of propositions and types, is the same as having the two maps $A\alpha^+$ and $A\alpha^-$.

These conditions are quite strong. They imply that we have a line between u and $A\alpha^+ u$ if $u \in A\alpha(0)$. This means that we have an element γ in $\prod_{i \in I} A\alpha(i)$ such that $\gamma(0) = u$ and $\gamma(1) = A\alpha^+ u$. The reason is that we have a homotopy $h : I \times I \to \Gamma,\ h(i, j) = \alpha(i \wedge j)$ between $h(0) = \lambda j.\alpha(0)$ and $h(1) = \alpha$. We can then define $\gamma(i) = Ah(i)^+ u$. We write $A\alpha \uparrow u$ this line. (Similarly we can define a line $A\alpha \downarrow u$ between $A\alpha^- u$ and u.)

Once we have noticed this fact, it is easy to check that this fibrations having this transfer property are closed by dependent sums and dependent products. This follows from the formulas

$$(\Sigma\ A\ B)\alpha^+(u, v) = (A\alpha^+ u, B(\alpha, A\alpha \uparrow u)^+ v)$$

and

$$(\Pi\ A\ B)\alpha^+ f\ u = B(\alpha, A\alpha \downarrow u)^+(f\ (A\alpha^-\ u))$$

This gives a model of type theory. We can extend it to type theory with one universe.

The following result will be used for interpreting the identity type. Notice that the proof uses that fact that the description axiom holds for simplicial sets.

Lemma 2.1 *Given a dependent family $A\sigma$ over Γ having the transfer property, if α is in Γ^{Δ_n} and u is in $A\alpha(i_0)$ for some i_0 in Δ_n then there exists $\tilde{u}$ in $\prod_{i \in \Delta_n} A\alpha(i)$ such that $\tilde{u}(i_0) = u$.*

Proof. We give the proof for $n = 1$. We consider the maps $\beta(j) = \alpha(i_0 \wedge j)$ and $\delta(j) = \alpha(i_0 \vee j)$. We define then $\tilde{u}(j) = A\beta \downarrow u(j)$ for $j \leqslant i_0$ and $\tilde{u}(j) = A\delta \uparrow u(j)$ for $j \geqslant i_0$. □

However we do not have a model of the identity type yet. We would like to say that a simplical set X has an identity type if the map $X^I \to X \times X$ has the transfer property. If we unfold what it means we get that we can extend any map $f(i,j) \in X$ defined for i,j in $(\{0\} \times I) \cup I \times \partial I$ to $f(1,j) \in X$ for all $j \in I$. Furthermore we should have $f(1,j) = f(0,j)$ if we have $f(i,j) = f(0,j)$ for all i,j in $I \times \partial I$.

The corresponding notion of constructive Kan fibration is the following. We introduce first the notation $\mathsf{B}(X) = (\{0\} \times \Delta_n) \cup I \times X$ if $X \to \Delta_n$ is a subobject of Δ_n. We define when a dependent simplicial set $A\sigma$ over $\sigma \in \Gamma$ has the *n-transfer property*. This means that, given σ_{ij} for i,j in $I \times \Delta_n$, we can extend any section $a(i,j) \in A\sigma_{ij}$ defined for i,j in $\mathsf{B}(\partial\Delta_n)$ to $a(1,j) \in A\sigma_{1j}$ for all j in Δ_n. Furthermore we should have $a(1,j) = a(0,j)$ if we have $\sigma_{ij} = \sigma_{0j}$ for all $j \in \Delta_n$ and $a(i,j) = a(0,j)$ for all i,j in $I \times \partial\Delta_n$. Intuitively an object a of shape $\mathsf{B}(\partial\Delta_n)$ looks like an open box and the extension $a(1)$ looks like a possible top for this box.

Theorem 2.2 *If $\Gamma \vdash A$ has the transfer property and $\Gamma.A \vdash B$ has the n-transfer property then $\Gamma \vdash \Pi\ A\ B$ has the n-transfer property. If both $\Gamma \vdash A$ and $\Gamma.A \vdash B$ have the n-transfer property then so does $\Gamma \vdash \Sigma\ A\ B$.*

Proof. We assume give σ_{ij} for i,j in $I \times \Delta_n$. We assume also give $v(i,j) \in (\Pi\ A\ B)\sigma_{ij}$ for i,j in $\mathsf{B}(\partial\Delta_n)$. We want then to extend $v(i,j)$ for (i,j) in $\{1\} \times \Delta_n$. Using Lemma 2.1, it is enough to define $v(1,j)\ w(1,j) \in B(\sigma_{1j}, w(1,j))$ for a line $w(1,j) \in A\sigma_{1j}$. We extend $w(1,j)$ to $w(i,j) \in A\sigma_{ij}$ for all $(i,j) \in I \times \Delta_n$ by defining

$$w(i,j) = A(\lambda i.\sigma_{ij}) \downarrow w(1,j)(i)$$

We have then $v(i,j)\ w(i,j) \in B(\sigma_{ij}, w(i,j))$ for i,j in $\mathsf{B}(\partial\Delta_n)$. By hypothesis on B this can be extended to a line $v(i,j)\ w(i,j)$ for $i = 1$ and $j \in I$. $\square$

Given $\Gamma \vdash A$ and $\Gamma \vdash a_0 : A$ and $\Gamma \vdash a_1 : A$ we interpret $\Gamma \vdash \mathsf{Id}_A\ a_0\ a_1$ by taking $(\mathsf{Id}_A\ a_0\ a_1)\sigma$ to be the simplical set of lines α in $(A\sigma)^I$ such that $\alpha(0) = a_0\sigma$ and $\alpha(1) = a_1\sigma$. Then $\Gamma \vdash \mathsf{Id}_A\ a_0\ a_1$ has the n-transfer property whenever $\Gamma \vdash A$ has the $(n+1)$-transfer property. If $\Gamma, x_0 : A, x_1 : A, p : \mathsf{Id}_A\ x_0\ x_1 \vdash C$ has the transfer property we get an interpretation of the elimination rule for the identity type since there is a homotopy between the constant line $\eta_0\ x_0$ and any line $\alpha : x_0 \to x_1$.

This means that we get a model of type theory with identity type by the fibrations that have the n-transfer property for all n.

3 A symbolic formulation

The goal of this section is to explain how we can implement the model described in the previous section, using the technique presented in [2, 3, 5].

We have projections $n+1$ maps $s_i : \Delta_{n+1} \to \Delta_n$ and $n \leqslant m$ we have maps $\eta : \Delta_m \to \Delta_n$ that can be described symbolically as a sequence of compositions $s_{i_1} \circ \ldots \circ s_{i_{m-n}}$ with $i_1 < \ldots < i_m$. By composition this defines a map $\eta : X^{\Delta_n} \to X^{\Delta_m}$. If we pull-back the mono $\partial\Delta_n \to \Delta_n$ along a projection $s : \Delta_m \to \Delta_n$ we get a mono $\partial_s\Delta_m \to \Delta_m$ which will be used in the symbolic description of the model.

3.1 Transfer property

We consider a combinatory algebra where objects can have different shapes. They are points, lines, triangles, ... but also squares, prisms, ... We can always apply an object of a given shape to another of the same shape.

We can think of this combinatory algebra as a simplicial set D. If u is an object of shape Δ_n we can consider $\eta_i\ u$ of shape Δ_{n+1} for $i \leqslant n$ and, if $n > 0$, we consider $d_i\ u$ of shape Δ_{n-1} for $i \leqslant n$. We also have elements of shape $I \times \Delta_n$ that are $(n+1)$-uple of elements of shape Δ_{n+1}. If we have an element u of shape $I \times X$ we can think of u as a function $I \times X \to D$ and we can consider $u(0)$ and $u(1)$ elements of shape X.

If u is an object of shape X and f is a map $f : Y \to X$ we write f^*u the object of shape Y obtained by composing u, seen as an element of D^X, and f. In the particular case where f is a mono, f^*u can be seen as the restriction of u to Y. For instance if X is the interval I, and u is a line, and $f : 1 \to I$ is $f(0) = 0$ then f^*u is $d_0\ u$.

Some objects in D are thought of as types. If α and β are of the same shape X, and α is a type and $\beta(u)$ is a type for $u : \alpha$ then $\Pi\ \alpha\ \beta$ and $\Sigma\ \alpha\ \beta$ are two types of shape X.

Given a type α of shape $I \times \Delta_n$, we associate then α^+ an object of shape Δ_n such that $\alpha^+\delta : \alpha(1)$ if $\delta : \alpha(0)$. Furthermore, if α is constant then $\alpha^+\delta = \delta$. We also have the following compatibility conditions: if $s : \Delta_m \to \Delta_n$ then $s^*(\alpha^+u) = ((I \times s)^*\alpha)^+(s^*u)$.

It follows from this that if we have a type α of shape $I \times Z$ then we can define α^+ object of shape Z such that $\alpha^+\delta : \alpha(1)$ if $\delta : \alpha(0)$. We have the compatibility conditions: if $f : T \to Z$ is any map then $f^*(\alpha^+u) = ((I \times f)^*\alpha)^+(f^*u)$. Furthermore, if α does not depend on its first argument then $\alpha^+\delta = \delta$.

We use these compatibility conditions to build an object $\alpha \uparrow \delta$ of shape $I \times X$ such that $\alpha \uparrow \delta(0) = \delta$ and $\alpha \uparrow \delta(1) = \alpha^+\delta$ for any type α of shape $I \times X$ and any object δ of shape X and of type $\alpha(0)$. For this we define first an object h of type $I \times (I \times X)$ such that $h(i, 0, x) = \alpha(0, x)$ and $h(i, 1, x) = \alpha(i, x)$. If ξ is the object of shape $I \times X$ which is such that $\xi(i) = \delta$ we can consider $h^+\xi$ which is an object of shape $I \times X$. By the compatibility conditions we have $h^+\xi(0) = \delta$ and $h^+\xi(1) = \alpha^+\delta$.

We can then define as in the previous section, but now interpreted symbolically

$$(\Sigma\ \alpha\ \beta)^+(u, v) = (\alpha^+u, \beta(\alpha \uparrow u)^+v)$$

and

$$(\Pi\ \alpha\ \beta)^+f\ u = \beta(\alpha \downarrow u)^+(f\ (\alpha^-\ u))$$

3.2 Identity type

For each mono $X \to \Delta_m$ which is obtained by pulling-back the mono $\partial\Delta_n \to \Delta_n$ along a projection $\Delta_m \to \Delta_n$ and for each type α of shape $I \times \Delta_m$, and δ is of shape $\mathsf{B}(X) = (\{0\} \times \Delta_m) \cup I \times X$, we introduce $\alpha^+_X\delta$ which is an object of shape Δ_m. Furthermore $\alpha^+_X\delta = \delta$ if $\alpha(i) = \alpha(0)$ and $\delta(i) = \delta(0)$ for all i in I. If X is empty we recover the transfer operation $\alpha^+\delta$. Along any map $s : \Delta_l \to \Delta_m$ the subobject X becomes $s^*X \to \Delta_l$ and we have a corresponding map $\mathsf{B}(s) : \mathsf{B}(s^*X) \to \mathsf{B}(X)$. The compatibility condition is then

$$s^*(\alpha^+_X\delta) = ((I \times s)^*\alpha)^+_{s^*X}(\mathsf{B}(s)^*\delta)$$

From this we deduce an operation $\alpha \uparrow_X \delta$ which is an object of shape $I \times \Delta_n$ and is a homotopy between $\delta(0)$ and $\alpha^+_X\delta$. We can also define $\alpha \Uparrow_X \delta$ which is an object of shape $\mathsf{B}(X)$ obtained by restricting $\alpha \uparrow_X \delta$. We can then define

$$(\Pi\ \alpha\ \beta)^+_X f\ u = \beta(\alpha \downarrow u)^+_X(f\ (\alpha \Downarrow u))$$

and

$$(\Sigma\ \alpha\ \beta)^+_X(u, v) = (\alpha^+_X u, \beta(\alpha \uparrow_X u)^+_X v)$$

3.3 The model

If α is an object of a given shape, and t is a term, then $t\alpha$ will be an object of the same shape. The main laws are

$$(t_1\ t_0)\alpha = t_1\alpha\ (t_0\alpha) \qquad ((\lambda t)\alpha\ u = t(\alpha, u)$$

$$(\Sigma\ A\ B)\alpha = \Sigma\ (A\alpha)\ (\lambda B)\alpha \qquad (\Pi\ A\ B)\alpha = \Pi\ (A\alpha)\ (\lambda B)\alpha$$

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