Cubical Type Theory

Free bounded distributive lattice

The free distributive lattice on a set J can be described as the set of finite antichains in the poset of finite subsets of J, for the order $L \leqslant M$ if, and only if, for all X in L there exists Y in M such that $Y \subseteq X$. We think of an element L as a formal representation of $\vee_{X\in L} \wedge_{u\in X} u$.

The free distributive lattice on J where we impose some relations of the form $\wedge_{u\in X} u = 0$ for some given set C of finite subsets has a similar description: it is the set of finite antichains of finite subsets not containing any element of C.

In both cases the elements of the form $\wedge_{u\in X} u$ are exactly the *join irreducible* element of the lattice, and we call $\wedge_{u\in X} u$ a *face* of an element L for X in L (these are exactly the maximal join irreducible element below this given element of the lattice).

Interval and Face lattice

$$r, s \ ::= \ 0 \mid 1 \mid i \mid 1-i \mid r \wedge s \mid r \vee s \qquad\qquad \varphi, \psi \ ::= \ 0 \mid 1 \mid (r=0) \mid (r=1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

The equality on the inverval $\mathbb{I}$ is the equality in the free bounded distributive lattice on generators $i, 1-i$. This lattice has a canonical involution, and hence a structure of de Morgan algebra. The equality in the face lattice $\mathbb{F}$ is the one for the free distributive lattice on formal generators $(i=0), (i=1)$ with the relation $(i=0) \wedge (i=1) = 0$. We have $[(r \vee s) = 1] = (r=1) \vee (s=1)$ and $[(r \wedge s) = 1] = (r=1) \wedge (s=1)$. An irreducible element of this lattice is a *face*, a conjunction of elements $(i=0)$ and $(j=1)$ and any element is a disjunction of irreducible elements (unique up to the absorption law).

The following observation will be useful for defining composition for glueing. Any formula φ has a decomposition $\delta \vee (\varphi_0 \wedge (i=0)) \vee (\varphi_1 \wedge (i=1))$ where δ is the disjunction of all faces of φ not containing i, and φ_0 (resp. φ_1) the disjunction of all faces α such that $\alpha \wedge (i=0)$ (resp. $\alpha \wedge (i=1)$) is a face of φ. We can then define $\forall i.\varphi$ as being δ.

Contexts and Terms

$$\begin{array}{lll}
\Delta, \Gamma & ::= & () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} \mid \Gamma, \varphi \\
t, u, A, B & ::= & x \mid \lambda x : A.t \mid t\ t \mid t\ r \mid \langle i \rangle t \mid (x : A) \to B \mid (x : A, B) \mid t, t \mid t.1 \mid t.2 \mid pt \\
pt & ::= & \psi_1 u_1 \vee \cdots \vee \psi_k u_k
\end{array}$$

We define ordinary substitution $t(x=u)$ and name susbtitution $t(i=r)$ as meta-operations as usual. We may write $t(i0)$ instead of $t(i=0)$ and $t(i1)$ instead of $t(i=1)$.

Basic typing rules

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash} \qquad \frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \qquad \frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (r=1) : \mathbb{F}} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (r=0) : \mathbb{F}}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash x : A}\ (x{:}A\ in\ \Gamma) \qquad \frac{\Gamma \vdash}{\Gamma \vdash i : \mathbb{I}}\ (i{:}\mathbb{I}\ in\ \Gamma)$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\ t : (x : A) \to B} \qquad \frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(u)}$$

Sigma types

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A, B)} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B(a)}{\Gamma \vdash (a, b) : (x : A, B)} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.1 : A} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.2 : B(z.1)}$$

Path types

$$\frac{\Gamma \vdash A \quad \Gamma \vdash a_0 : A \quad \Gamma \vdash a_1 : A}{\Gamma \vdash \mathsf{Path}\ A\ a_0\ a_1} \qquad \frac{\Gamma \vdash A \quad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathsf{Path}\ A\ t(i0)\ t(i1)}$$

$$\frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1 \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\ r : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 1 = a_1 : A}$$

We define $1_a : \mathsf{Path}\ A\ a\ a$ as $1_a = \langle i \rangle a$.

We add the usual β and η-conversion laws, as well as projection laws and surjective pairing.

With these rules we also can justify function extensionality

$$\frac{\Gamma \vdash t : (x : A) \to B \quad \Gamma \vdash u : (x : A) \to B \quad \Gamma \vdash p : (x : A) \to \mathsf{Path}\ B\ (t\ x)\ (u\ x)}{\Gamma \vdash \langle i \rangle \lambda x : A.\ p\ x\ i : \mathsf{Path}\ ((x : A) \to B)\ t\ u}$$

We also can justify the fact that any element in $(x : A, \mathsf{Path}\ A\ a\ x)$ is equal to $(a, 1_a)$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : A \quad \Gamma \vdash p : \mathsf{Path}\ A\ a\ b}{\Gamma \vdash \langle i \rangle (p\ i, \langle j \rangle p\ (i \wedge j)) : \mathsf{Path}\ (x : A, \mathsf{Path}\ A\ a\ x)\ (a, 1_a)\ (b, p)}$$

For justifying the transitivity of equality, we need A to have *composition operations*.

Partial elements

$$\frac{\Gamma \vdash \varphi \leqslant \psi \quad \Gamma, \psi \vdash A}{\Gamma, \varphi \vdash A} \qquad \frac{\Gamma \vdash \varphi \leqslant \psi \quad \Gamma, \psi \vdash u : A}{\Gamma, \varphi \vdash u : A}$$

$$\frac{\Gamma, \psi_1 \vdash A_1 \quad \ldots \quad \Gamma, \psi_k \vdash A_k \quad \Gamma, \psi_i \wedge \psi_j \vdash A_i = A_j}{\Gamma, \psi_1 \vee \cdots \vee \psi_k \vdash \psi_1 A_1 \vee \cdots \vee \psi_k A_k}$$

$$\frac{\Gamma, \psi_1 \vdash u_1 : A_1 \quad \ldots \quad \Gamma, \psi_k \vdash u_k : A_k \quad \Gamma, \psi_i \wedge \psi_j \vdash A_i = A_j \quad \Gamma, \psi_i \wedge \psi_j \vdash u_i = u_j : A_i}{\Gamma, \psi_1 \vee \cdots \vee \psi_k \vdash \psi_1 u_1 \vee \cdots \vee \psi_k u_k : \psi_1 A_1 \vee \cdots \vee \psi_k A_k}$$

We can have $k = 0$ in which case we get a dummy element of type A in the context $\Gamma, 0$.

We also have $\psi_1 u_1 \vee \cdots \vee \psi_k u_k = u_i : A$ if $\psi_i = 1$ and $\psi_1 \vee \cdots \vee \psi_k \vdash u = v : A$ if $\psi_i \vdash u = v : A$ for $i = 1, \ldots, k$. Finally, we add that $\Gamma \vdash r = 1$ if $\Gamma \vdash 1 = (r = 1)$.

If $\Gamma, \varphi \vdash u : A$ then $\Gamma \vdash a : A[\varphi \mapsto u]$ is an abreviation for $\Gamma \vdash a : A$ and $\Gamma, \varphi \vdash a = u : A$. In this case, we see this element a as a witness that the partial element u, defined on the extent φ, is *connected*.

For instance if $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I}, \varphi \vdash u : A$ where $\varphi = (i = 0) \vee (i = 1)$ then the element u is determined by two element $\Gamma \vdash a_0 : A(i0)$ and $\Gamma \vdash a_1 : A(i1)$ and an element $\Gamma, i : \mathbb{I} \vdash a : A[\varphi \mapsto u]$ gives a path connecting a_0 and a_1.

We may write $\Gamma \vdash a : A[\psi_1 \mapsto u_1, \ldots, \psi_k \mapsto u_k]$ for $\Gamma \vdash a : A[\psi_1 \vee \cdots \vee \psi_k \mapsto \psi_1 u_1 \vee \cdots \vee \psi_k u_k]$. This means that $\Gamma \vdash a : A$ and $\Gamma, \psi_i \vdash a = u_i : A$ for $i = 1, \ldots, k$.

Composition operation

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

Kan filling operation

We recover Kan filling operation

$$\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 = \mathsf{comp}^j\ A(i \wedge j)\ [\varphi \mapsto u(i \wedge j), (i = 0) \mapsto a_0]\ a_0 : A$$

The element $i : \mathbb{I} \vdash v = \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 : A$ satisfies

$$\Gamma \vdash v(i0) = a_0 : A(i0) \qquad \Gamma \vdash v(i1) = \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1) \qquad \Gamma, \varphi, i : \mathbb{I} \vdash v = u : A$$

Recursive definition of composition

The operation $\mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0$ is defined by induction on A.

Product type

In the case of a product type $i : \mathbb{I} \vdash (x : A) \to B = C$, we have $i : \mathbb{I}, \varphi \vdash \mu : C$ with and $\vdash \lambda_0 : C(i0)[\varphi \mapsto \mu(i0)]$ and we define, for $\vdash u_1 : A(i1)$

$$(\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0)\ u_1 = \mathsf{comp}^i\ B(x = v)\ [\varphi \mapsto \mu\ v]\ (\lambda_0\ u_0)$$

where $i : \mathbb{I} \vdash v = w(1 - i) : A$ and $i : \mathbb{I} \vdash w = \mathsf{fill}^i\ A(1 - i)\ []\ u_1 : A(1 - i)$ and $u_0 = v(i0) : A(i0)$.

Path type

In the case of path type $i : \mathbb{I} \vdash \mathsf{Path}\ A\ u\ v = C$ we have $i : \mathbb{I}, \varphi \vdash \mu : C$ and $\vdash p_0 : C(i0)[\varphi \mapsto \mu(i0)]$. We define

$$\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ p_0 = \langle j \rangle \mathsf{comp}^i\ A\ [\varphi \mapsto \mu\ j, (j = 0) \mapsto u, (j = 1) \mapsto v]\ (p_0\ j)$$

Sum type

In the case of a sigma type $i : \mathbb{I} \vdash (x : A, B) = C$ given $i : \mathbb{I}, \varphi \vdash w : C$ and $\vdash w_0 : C(i0)[\varphi \mapsto w(i0)]$ we define

$$\mathsf{comp}^i\ C\ [\varphi \mapsto w]\ w_0 = (\mathsf{comp}^i\ A\ [\varphi \mapsto w.1]\ w_0.1, \mathsf{comp}^i\ B(x = a)\ [\varphi \mapsto w.2]\ w_0.2)$$

where $i : \mathbb{I} \vdash a = \mathsf{fill}^i\ A\ [\varphi \mapsto w.1]\ w_0.1 : A$.

Example

If $i : \mathbb{I} \vdash A$, composition for $\varphi = 0$ corresponds to a transport function $A(i0) \to A(i1)$.

If I is an object of $\mathcal{C}$ the lattice $\mathbb{F}(I)$ has a greatest element < 1 which is the disjunction of all $(i = 0) \vee (i = 1)$ for i in I. This element can be called the *boundary* of I. Composition w.r.t. this boundary gives the usual operation of Kan composition, which witnesses the existence of a lid for any open box.

Two derived operations

The first derived operation states that the image of a composition is path equal to the composition of the respective images.

Lemma 0.1 *If we have $\Delta, i : \mathbb{I} \vdash \sigma : T \to A$, $\Delta \vdash \psi$ and $\Delta, \psi, i : \mathbb{I} \vdash t : T$ with $\Delta \vdash t_0 : T(i0)[\psi \mapsto t(i0)]$ then we can build*

$$\Delta \vdash \mathsf{pres}(\sigma, [\psi \mapsto t], t_0) : \mathsf{Path}\ A(i1)\ (\mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0)\ \sigma(i1)\ (\mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0)$$

where $\Delta \vdash a_0 = \sigma(i0)\ t_0 : A(i0)$ and $\Delta, i : \mathbb{I}, \psi \vdash a = \sigma\ t : A$. Furthermore, we have

$$\Delta, \psi \vdash \mathsf{pres}(\sigma, [\psi \mapsto t], t_0) = \langle j \rangle a(i1)$$

We define $\mathsf{isContr}\ A = (x : A, (y : A) \to \mathsf{Path}\ A\ x\ y)$ and $\mathsf{isEquiv}\ A\ B\ f = (y : B) \to \mathsf{isContr}(x : A, \mathsf{Path}\ A\ y\ (f\ x))$ and $\mathsf{Equiv}(T, A) = (f : T \to A, \mathsf{isEquiv}\ T\ A\ f)$.

The second operation corresponds to a reformulation of the notion of being contractible.

Lemma 0.2 *We have an operation*

$$\frac{\Gamma \vdash p : \mathsf{isContr}\ A \qquad \Gamma, \varphi \vdash u : A}{\Gamma \vdash \mathsf{ext}\ p\ [\varphi \mapsto u] : A[\varphi \mapsto u]}$$

and it follows that we have an operation $\mathsf{equiv}(\sigma, [\delta \mapsto t], a) = \mathsf{ext}\ (\sigma.2\ a)\ [\delta \mapsto (t, \langle j \rangle a)]$

$$\frac{\Delta \vdash \sigma : \mathsf{Equiv}(T, A) \qquad \Delta, \delta \vdash t : T \qquad \Delta \vdash a : A[\delta \mapsto \sigma\ t]}{\Delta \vdash \mathsf{equiv}(\sigma, [\delta \mapsto t], a) : (x : T, \mathsf{Path}\ A\ a\ (\sigma\ x))[\delta \mapsto (t, \langle j \rangle a)]}$$

A definition of ext

We assume given $\Gamma \vdash p : \mathsf{isContr}\ A$ and $\Gamma, \varphi \vdash u : A$. We define $\mathsf{ext}\ p\ [\varphi \mapsto u] = \mathsf{comp}^i\ A\ [\varphi \mapsto p.2\ u\ i]\ p.1$ so that $\Gamma \vdash \mathsf{ext}\ p\ [\varphi \mapsto u] : A[\varphi \mapsto u]$.

A definition of pres

We assume given $\Delta, i : \mathbb{I} \vdash \sigma : T \to A$, $\Delta \vdash \psi$ and $\Delta, \psi, i : \mathbb{I} \vdash t : T$ with $\Delta \vdash t_0 : T(i0)[\psi \mapsto t(i0)]$. We define $\Delta \vdash a_0 = \sigma(i0)\ t_0 : A(i0)$ and $\Delta, i : \mathbb{I}, \psi \vdash a = \sigma\ t : A$, and

$$\Delta, i : \mathbb{I} \vdash u = \mathsf{fill}^i\ A\ [\psi \mapsto a]\ a_0 : A \qquad \Delta, i : \mathbb{I} \vdash v = \mathsf{fill}^i\ T\ [\psi \mapsto t]\ t_0 : T$$

We define then $\mathsf{pres}(\sigma, [\psi \mapsto t], t_0) = \langle j \rangle \mathsf{comp}^i\ A\ [\psi \mapsto \sigma\ t, (j = 0) \mapsto \sigma\ v, (j = 1) \mapsto u]\ a_0$

Glueing

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash \sigma : \mathsf{Equiv}(T, A)}{\Gamma \vdash \mathsf{Glue}(A, [\varphi \mapsto (T, \sigma)])} \varphi \neq 1$$

$$\frac{\Gamma, \varphi \vdash \sigma : \mathsf{Equiv}(T, A) \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto \sigma t]}{\Gamma \vdash \mathsf{Glue}(a, [\varphi \mapsto t]) : \mathsf{Glue}(A, [\varphi \mapsto (T, \sigma)])} \varphi \neq 1$$

We define $\mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) = \mathsf{Glue}(A, [\varphi \mapsto (T, \sigma)])$ if $\varphi \neq 1$ and $\mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) = T$ if $\varphi = 1$. Similarly we define $\mathsf{glue}(a, [\varphi \mapsto t]) = \mathsf{Glue}(a, [\varphi \mapsto t])$ if $\varphi \neq 1$ and $\mathsf{glue}(a, [\varphi \mapsto t]) = t$ if $\varphi = 1$.

Any element of the type $\mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])$ can be written in an unique way of the form $\mathsf{glue}(a, [\varphi \mapsto t])$ with $\varphi \vdash t : T$ and $a : A[\varphi \mapsto \sigma t]$.

We define the substitution $\mathsf{Glue}(A, [\varphi \mapsto (T, \sigma)])f = \mathsf{glue}(Af, [\varphi f \mapsto (Tf, \sigma f)])$ and $\mathsf{Glue}(a, [\varphi \mapsto t])f = \mathsf{glue}(af, [\varphi f \mapsto tf])$.

Composition for glueing

Assume $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I} \vdash \varphi$ and $\Gamma, i : \mathbb{I}, \varphi \vdash \sigma : \mathsf{Equiv}(T, A)$. We write $B = \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])$. Assume also $\Gamma \vdash \psi$ and $\Gamma, i : \mathbb{I}, \psi \vdash b = \mathsf{glue}(a, [\varphi \mapsto t]) : B$ and $\Gamma \vdash b_0 = \mathsf{glue}(a_0, [\varphi(i0) \mapsto t_0]) : B(i0)[\psi \mapsto b(i0)]$.

The goal is to build $\Gamma \vdash b_1 : B(i1)[\psi \mapsto b(i1)]$. Furthermore, we should have $b_1 = \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0$ if $\Gamma, i : \mathbb{I} \vdash \varphi = 1$.

We have $\Gamma, \psi \vdash a(i0) = a_0 : A(i0)$ and $\Gamma, \psi \wedge \varphi(i0) \vdash t(i0) = t_0 : T(i0)$. Furthermore $\Gamma, \varphi(i0) \vdash a_0 = \sigma(i0)t_0 : A(i0)$ and $\Gamma, i : \mathbb{I}, \varphi \wedge \psi \vdash a = \sigma t : A$.

We define $a'_1 = \mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0$ so that $\Gamma \vdash a'_1 : A(i1)$ and $\Gamma, \psi \vdash a'_1 = a(i1) : A(i1)$.
Take $\delta = \forall i.\varphi$. We have $\Gamma, \delta, \psi, i : \mathbb{I} \vdash a = \sigma\ t$ and $\Gamma, \delta \vdash a_0 = \sigma(i0)\ t_0$. Hence, using Lemma 0.1

$$\Gamma, \delta \vdash \omega = \mathsf{pres}\ \sigma\ [\psi \mapsto t]\ t_0 : \mathsf{Path}\ A(i1)\ a'_1\ (\sigma(i1)\ t'_1)$$

where $t'_1 = \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0$. We can then define $a''_1 = \mathsf{comp}^j\ A(i1)\ [\delta \mapsto \omega\ j, \psi \mapsto a(i1)]\ a'_1$ so that $\Gamma \vdash a''_1 : A(i1)$ and $\Gamma, \psi \vdash a''_1 = a(i1) : A(i1)$ and $\Gamma, \delta \vdash a''_1 = \sigma(i1)\ t'_1 : A(i1)$.

We have $\Gamma, \varphi(i1) \vdash \sigma(i1) : T(i1) \to A(i1)$ and $\Gamma \vdash a''_1 : A(i1)$ and $\Gamma, \delta \vdash a''_1 = \sigma(i1)\ t'_1$ and $\Gamma, \psi \wedge \varphi(i1) \vdash a''_1 = a(i1) = \sigma(i1)\ t(i1)$. Using Lemma 0.2 we get

$$t_1 = \mathsf{equiv}(\sigma(i1), [\delta \mapsto t'_1, \psi \mapsto t(i1)], a''_1).1 \qquad \alpha = \mathsf{equiv}(\sigma(i1), [\delta \mapsto t'_1, \psi \mapsto t(i1)], a''_1).2$$

so that $\Gamma, \varphi(i1) \vdash t_1 : T(i1)$ and $\Gamma, \varphi(i1) \vdash \alpha : \mathsf{Path}\ A(i1)\ a''_1\ (\sigma(i1)\ t_1)$. We then define

$$a_1 = \mathsf{comp}^j\ A(i1)\ [\varphi(i1) \mapsto \alpha\ j, \psi \mapsto a(i1)]\ a''_1 \qquad b_1 = \mathsf{glue}(a_1, [\varphi(i1) \mapsto t_1])$$

We have $\Gamma \vdash b_1 : B(i1)[\psi \mapsto b(i1)]$ as required and, if $\Gamma, i : \mathbb{I} \vdash \varphi = 1$ we have $b_1 = \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0$.

Identity types

We explain how to define an identity type with the required computation rule, following an idea due to Andrew Swan.

We define a new type $\mathsf{Id}\ A\ a_0\ a_1$ with the introduction rule

$$\frac{\Gamma \vdash \omega : \mathsf{Path}\ A\ a_0\ a_1[\varphi \mapsto \langle i \rangle a_0]}{\Gamma \vdash (\omega, \varphi) : \mathsf{Id}\ A\ a_0\ a_1}$$

We can now define $\mathsf{r}(a) = (\langle j \rangle a, 1) : \mathsf{Id}\ A\ a\ a$.

Given $\Gamma \vdash \alpha = (\omega, \varphi) : \mathsf{Id}\ A\ a\ x$ we define $\Gamma, i : \mathbb{I} \vdash \alpha^*(i) : \mathsf{Id}\ A\ a\ (\alpha\ i)$

$$\alpha^*(i) = (\langle j \rangle \omega(i \wedge j), \varphi \vee (i = 0))$$

This is well defined since $\Gamma, i : \mathbb{I}, (i = 0) \vdash \langle j \rangle \omega(i \wedge j) = \langle j \rangle a$ and $\Gamma, i : \mathbb{I}, \varphi \vdash \langle j \rangle \omega(i \wedge j) = \langle j \rangle a$.

If we have $\Gamma, x : A, \alpha : \mathsf{Id}\ A\ a\ x \vdash C$ and $\Gamma \vdash b : A$ and $\Gamma \vdash \beta : \mathsf{Id}\ A\ a\ b$ and $\Gamma \vdash d : C(a, \mathsf{r}(a))$ we take, for $\beta = (\omega, \varphi)$

$$J\ b\ \beta\ d = \mathsf{comp}^i\ C(\omega\ i, \beta^*(i))\ [\varphi \mapsto d]\ d : C(b, \beta)$$

and we have $J\ a\ \mathsf{r}(a)\ d = d$ as desired.

If $i : \mathbb{I} \vdash \mathsf{Id}\ A\ a\ b$ and $p_0 = (\omega_0, \psi_0) : \mathsf{Id}\ A(i0)\ a(i0)\ b(i0)$ and $\varphi, i : \mathbb{I} \vdash q = (\omega, \psi) : \mathsf{Id}\ A\ a\ b$ such that $\varphi \vdash q(i0) = p_0$ we define $\mathsf{comp}^i\ (\mathsf{Id}\ A\ a\ b)\ [\varphi \mapsto q]\ p_0$ to be $(\gamma, \varphi \wedge \psi(i1))$ where

$$\gamma = \langle j \rangle \mathsf{comp}^i\ A\ [\varphi \mapsto \omega\ j, (j = 0) \mapsto a, (j = 1) \mapsto b]\ (\omega_0\ j)$$

Factorization

The same idea of Andrew Swan can be used to factorize a map

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \mathsf{G}(f)} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma, \varphi \vdash a : A \qquad \Gamma \vdash b : B[\varphi \mapsto f\ a]}{\Gamma \vdash (b, [\varphi \mapsto a]) : \mathsf{G}(f)}$$

We define $p_G : \mathsf{G}(f) \to B$ by $p_G(b, [\varphi \mapsto a]) = b$ and $\mathsf{c}(a) = (f\ a, [1 \mapsto a])$ and we have a factorization of the map $f = p_G \circ \mathsf{c}$.

The composition for $G(f)$ is defined by

$$\mathsf{comp}^i\ G(f)\ [\varphi \mapsto (b, [\psi \mapsto a])]\ (b_0, [\psi_0 \mapsto a_0]) = (\mathsf{comp}^i\ B\ [\varphi \mapsto b]\ b_0, [\varphi \wedge \psi(i1) \mapsto a(i1)])$$

Here is one application of the type $G(f)$. Suppose that we have a dependent type $D(v)\ (v : B)$ with a section $g(v) : C(v)\ (v : B)$ and $h(a) : C(f\ a)\ (a : A)$ with $\omega(a) : \mathsf{Path}\ C(f\ a)\ g(f\ a)\ h(a)\ (a : A)$. We can define a new section $\tilde{g}(u) : C(p_G\ u)\ (u : \mathsf{G}(f))$ such that $\tilde{g}(\mathsf{c}\ a) = h(a)\ (a : A)$. The definition is

$$\tilde{g}(b, [\varphi \mapsto a]) = \mathsf{comp}^i\ C(b)\ g(b)\ [\varphi \mapsto \omega(a)\ i]$$

It can be checked that c has the lifting property w.r.t. any trivial fibrations. Also p_G is a trivial fibration, since $G(f)$ can be defined as the sigma type $(b : B, T_f(b))$ where $T_f(b)$ is the contractible type of element $\varphi \mapsto a$ with $\Gamma, \varphi \vdash a : A$ and $\Gamma, \varphi \vdash f\ a = b : B$.

Appendix 1: self-contained operational semantics

We use $\alpha, \beta, \gamma, \dots$ for the "faces", irreducible elements of the distributive lattice $\mathbb{F}$. If we restrict context as follows

$$\Gamma \quad ::= \quad () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} \mid \Gamma, \alpha$$

then any partial element in such a context is equal to a total element. This follows from the fact that faces are irreducible element. To test a judgement in a context Γ, φ is then reduced to test the judgement in the context Γ, α for all irreducible component α of φ.

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash} \qquad \frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \qquad \frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash} \qquad \frac{\Gamma \vdash}{\Gamma \vdash x : A}\ (x{:}A\ \mathit{in}\ \Gamma) \qquad \frac{\Gamma \vdash}{\Gamma \vdash i : \mathbb{I}}\ (i{:}\mathbb{I}\ \mathit{in}\ \Gamma)$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\ t : (x : A) \to B} \qquad \frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(u)}$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A, B)} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B(a)}{\Gamma \vdash (a, b) : (x : A, B)} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.1 : A} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.2 : B(z.1)}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A}{\Gamma \vdash \mathsf{Path}\ A\ a_0\ a_1} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathsf{Path}\ A\ t(i0)\ t(i1)}$$

$$\frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1 \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\ r : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 1 = a_1 : A}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

$$\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 = \mathsf{comp}^j\ A(i \wedge j)\ [\varphi \mapsto u(i \wedge j), (i = 0) \mapsto a_0]\ a_0 : A$$

For $i : \mathbb{I} \vdash C = (x : A) \to B$

$$(\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0)\ u_1 = \mathsf{comp}^i\ B(x = v)\ [\varphi \mapsto \mu\ v]\ (\lambda_0\ u_0)$$

where $i : \mathbb{I} \vdash v = \mathsf{fill}^i\ A(1 - i)\ []\ u_1 : A$ and $u_0 = v(i0) : A(i0)$.

For $i : \mathbb{I} \vdash C = \mathsf{Path}\ A\ u\ v$

$$\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ p_0 = \langle j \rangle \mathsf{comp}^i\ A\ [\varphi \mapsto \mu\ j, (j = 0) \mapsto u, (j = 1) \mapsto v]\ (p_0\ j)$$

For $i : \mathbb{I} \vdash C = (x : A, B)$

$$\mathsf{comp}^i\ C\ [\varphi \mapsto w]\ w_0 = (\mathsf{comp}^i\ A\ [\varphi \mapsto w.1]\ w_0.1, \mathsf{comp}^i\ B(x = a)\ [\varphi \mapsto w.2]\ w_0.2)$$

where $i : \mathbb{I} \vdash a = \mathsf{fill}^i\ A\ [\varphi \mapsto w.1]\ w_0.1 : A$.

We define $\mathsf{isContr}\ A = (x : A, (y : A) \to \mathsf{Path}\ A\ x\ y)$ and $\mathsf{isEquiv}\ A\ B\ f = (y : B) \to \mathsf{isContr}(x : A, \mathsf{Path}\ A\ y\ (f\ x))$ and $\mathsf{Equiv}(T, A) = (f : T \to A, \mathsf{isEquiv}\ T\ A\ f)$.

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash \sigma : \mathsf{Equiv}(T, A)}{\Gamma \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) \qquad \Gamma, \varphi \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) = T}$$

$$\frac{\Gamma, \varphi \vdash \sigma : \mathsf{Equiv}(T, A) \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto \sigma t]}{\Gamma \vdash \mathsf{glue}(a, [\varphi \mapsto t]) : \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])[\varphi \mapsto t]}$$

For $\Gamma, i : \mathbb{I} \vdash B = \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])$ we define

$$\mathsf{comp}^i\ B\ [\psi \mapsto \mathsf{glue}(a, [\varphi \mapsto t])]\ \mathsf{glue}(a_0, [\varphi(i0) \mapsto t_0]) = \mathsf{glue}(a_1, [\varphi(i1) \mapsto t_1])$$

where

$$\begin{array}{lcll}
a_1 & = & \mathsf{comp}^j\ A(i1)\ [\varphi(i1) \mapsto \alpha\ j, \psi \mapsto a(i1)]\ a_1'' & \Gamma \\
t_1 & = & \mathsf{equiv}(\sigma(i1), [\delta \mapsto t_1', \psi \mapsto t(i1)], a_1'').1 & \Gamma, \varphi(i1) \\
\alpha & = & \mathsf{equiv}(\sigma(i1), [\delta \mapsto t_1', \psi \mapsto t(i1)], a_1'').2 & \Gamma, \varphi(i1) \\
a_1'' & = & \mathsf{comp}^j\ A(i1)\ [\delta \mapsto \omega\ j, \psi \mapsto a(i1)]\ a_1' & \Gamma \\
\omega & = & \mathsf{pres}\ \sigma\ [\psi \mapsto t]\ t_0 & \Gamma, \delta \\
t_1' & = & \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0 & \Gamma, \delta \\
a_1' & = & \mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0 & \Gamma \\
\delta & = & \forall i.\varphi & \Gamma
\end{array}$$

Name-free presentation

$$
\begin{array}{lcl}
\Gamma, \Delta & ::= & () \mid \Gamma.A \mid \Gamma.\mathbb{I} \mid \Gamma, \varphi \\
\varphi, \psi & ::= & 0 \mid 1 \mid (r = 0) \mid (r = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi f \\
r, s & ::= & 0 \mid 1 \mid \mathsf{q} \mid 1 - r \mid r \wedge s \mid r \vee s \mid rf \\
t, u, A, B & ::= & \mathsf{q} \mid \lambda t \mid \mathsf{app}(t, t) \mid t\ r \mid \langle\rangle t \mid \Pi\ A\ B \mid \Sigma\ A\ B \mid t, t \mid t.1 \mid t.2 \\
 & ::= & tf \mid \mathsf{comp}\ A\ [\varphi \mapsto u]\ t \mid \mathsf{Glue}(A, [\varphi \mapsto (T, u)]) \mid \mathsf{Glue}(a, [\varphi \mapsto u]) \mid pt \\
pt & ::= & \psi_1 u_1 \vee \cdots \vee \psi_k u_k \\
f, g & ::= & \mathsf{p} \mid gf \mid 1 \mid (f, u) \mid (f, r)
\end{array}
$$

$$
\frac{\Gamma \vdash A}{\Gamma.A \vdash} \qquad \frac{\Gamma \vdash}{\Gamma.\mathbb{I} \vdash} \qquad \frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash} \qquad \frac{\Gamma \vdash A}{\Gamma.A \vdash \mathsf{q} : A\mathsf{p}} \qquad \frac{\Gamma \vdash}{\Gamma.\mathbb{I} \vdash \mathsf{q} : \mathbb{I}}
$$

$$
\frac{\Gamma.A \vdash B}{\Gamma \vdash \Pi\ A\ B} \qquad \frac{\Gamma.A \vdash t : B}{\Gamma \vdash \lambda\ t : \Pi\ A\ B} \qquad \frac{\Gamma \vdash t : \Pi\ A\ B \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{app}(t, u) : B[u]}
$$

$$
\frac{\Gamma.A \vdash B}{\Gamma \vdash \Sigma\ A\ B} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[a]}{\Gamma \vdash (a, b) : \Sigma\ A\ B} \qquad \frac{\Gamma \vdash z : \Sigma\ A\ B}{\Gamma \vdash z.1 : A} \qquad \frac{\Gamma \vdash z : \Sigma\ A\ B}{\Gamma \vdash z.2 : B[z.1]}
$$

$$
\frac{\Gamma \vdash A \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A}{\Gamma \vdash \mathsf{Path}\ A\ a_0\ a_1} \qquad \frac{\Gamma \vdash A \qquad \Gamma.\mathbb{I} \vdash t : A}{\Gamma \vdash \langle\rangle t : \mathsf{Path}\ A\ t[0]\ t[1]}
$$

$$
\frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1 \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\ r : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 1 = a_1 : A}
$$

$$
\frac{\Gamma \vdash \varphi \qquad \Gamma.\mathbb{I} \vdash A \qquad \Gamma.\mathbb{I}, \varphi\mathsf{p} \vdash u : A \qquad \Gamma \vdash a_0 : A[0][\varphi \mapsto u[0]]}{\Gamma \vdash \mathsf{comp}\ A\ [\varphi \mapsto u]\ a_0 : A[1][\varphi \mapsto u[1]]}
$$

$$
\frac{\Gamma \vdash}{1 : \Gamma \to \Gamma} \qquad \frac{f : \Delta \to \Gamma \qquad g : \Theta \to \Delta}{fg : \Theta \to \Gamma} \qquad \frac{\Gamma \vdash A \qquad f : \Delta \to \Gamma}{\Delta \vdash Af} \qquad \frac{\Gamma \vdash t : A \qquad f : \Delta \to \Gamma}{\Delta \vdash tf : Af}
$$

$$
\frac{f : \Delta \to \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash u : A\sigma}{(f, u) : \Delta \to \Gamma.A} \qquad \frac{f : \Delta \to \Gamma \qquad \Delta \vdash r : \mathbb{I}}{(f, r) : \Delta \to \Gamma.\mathbb{I}}
$$

$$
\frac{f : \Delta \to \Gamma \qquad \Delta \vdash \psi : \mathbb{F}}{f : \Delta, \psi \to \Gamma} \qquad \frac{f : \Delta \to \Gamma \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Delta \vdash 1 = \varphi f}{f : \Delta \to \Gamma, \varphi}
$$

$$
1f = f1 = f \qquad (fg)h = f(gh) \qquad A1 = A \qquad (Af)g = A(fg) \qquad u1 = u \qquad (uf)g = u(fg)
$$

$$
(f, u)g = (fg, ug) \qquad \mathsf{p}(f, u) = f \qquad \mathsf{q}(f, u) = u \qquad (f, r)g = (fg, rg) \qquad \mathsf{p}(f, r) = f \qquad \mathsf{q}(f, r) = r
$$

$$
(\Pi\ A\ B)f = \Pi\ (Af)\ (B(f\mathsf{p}, \mathsf{q})) \qquad (\Sigma\ A\ B)f = \Sigma\ (Af)\ (B(f\mathsf{p}, \mathsf{q}))
$$

$$
\mathsf{app}(w, u)f = \mathsf{app}(wf, uf) \qquad \mathsf{app}(\lambda b, u) = b[u] \qquad w = \lambda(\mathsf{app}(w\mathsf{p}, \mathsf{q})) \qquad (\lambda b)f = \lambda(b(f\mathsf{p}, \mathsf{q}))
$$

$$
(t\ r)f = tf\ rf \qquad (\langle\rangle b)\ r = b[r] \qquad w = \langle\rangle(w\mathsf{p}\ \mathsf{q}) \qquad (\langle\rangle b)f = \langle\rangle b(f\mathsf{p}, \mathsf{q})
$$

$$
(t_0, t_1)f = (t_0 f, t_1 f) \qquad (u, v).1 = u \qquad (u, v).2 = v \qquad (\mathsf{p}, \mathsf{q}) = 1
$$

$$
(t.1)f = tf.1 \qquad (t.2)f = tf.2
$$

We have used the defined operation $[u] = (1, u)$

Appendix 2: spheres

We define S^1 by the rules.

$$\frac{}{\Gamma \vdash \mathsf{S}^1} \qquad \frac{}{\Gamma \vdash \mathsf{base} : \mathsf{S}^1} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{loop}(r) : \mathsf{S}^1} r \neq 0, 1$$

$$\frac{\Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{S}^1 \qquad \Gamma \vdash u_0 : \mathsf{S}^1[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{S}^1} \varphi \neq 1$$

We define the substitution $\mathsf{base}f = \mathsf{base}$ and $\mathsf{loop}(r)f = \mathsf{loop}(rf)$ if $rf \neq 0, 1$ and $\mathsf{loop}(r)f = \mathsf{base}$ if $rf = 0$ or 1.

Similarly we define $(\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0)f = \mathsf{hcomp}^j\ [\varphi f \mapsto u(f, i = j)]\ u_0 f$ if $\varphi f \neq 1$ and $(\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0)f = u(f, i = 1)$ if $\varphi f = 1$.

Using these operations, we can define

$$\frac{\Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{S}^1 \qquad \Gamma \vdash u_0 : \mathsf{S}^1[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{S}^1[\varphi \mapsto u(i1)]}$$

by $\mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 = \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0$ if $\varphi \neq 1$ and $\mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 = u(i1)$ if $\varphi = 1$.

We have a similar definition for S^n taking as constructors base and $\mathsf{loop}(r_1, \ldots, r_n)$, all $r_i \neq 0, 1$, with the substitution $\mathsf{loop}(r_1, \ldots, r_n)f = \mathsf{loop}(r_1 f, \ldots, r_n f)$ if all $r_i f$ are $\neq 0, 1$ and $\mathsf{loop}(r_1, \ldots, r_n)f = \mathsf{base}$ if some $r_i f$ is 0 or 1.

Appendix 3: propositional truncation

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{inh}\ A} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{inc}\ a : \mathsf{inh}\ A} \qquad \frac{\Gamma \vdash u_0 : \mathsf{inh}\ A \qquad \Gamma \vdash u_1 : \mathsf{inh}\ A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{squash}(u_0, u_1, r) : \mathsf{inh}\ A} r \neq 0, 1$$

$$\frac{\Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{inh}\ A \qquad \Gamma \vdash u_0 : \mathsf{inh}\ A[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{inh}\ A} \varphi \neq 1$$

The substitution is then $\mathsf{squash}(u_0, u_1, r)f = \mathsf{squash}(u_0 f, u_1 f, rf)$ if $rf \neq 0, 1$ and $\mathsf{squash}(u_0, u_1, r)f = u_0 f$ if $rf = 0$ and $\mathsf{squash}(u_0, u_1, r)f = u_1 f$ if $rf = 1$. Similarly we deifne $(\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0)f = \mathsf{comp}^j\ [\varphi f \mapsto u(f, i = j)]\ u_0 f$ if $\varphi f \neq 1$ and $(\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0)f = u(f, i = 1)$ if $\varphi f = 1$.

We can then define two operations

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash u_0 : \mathsf{inh}\ A(i0)}{\Gamma \vdash \mathsf{transp}\ u_0 : \mathsf{inh}\ A(i1)} \qquad \frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I} \vdash u : \mathsf{inh}\ A}{\Gamma, i : \mathbb{I} \vdash \mathsf{squeeze}\ u : \mathsf{inh}\ A(i1)}$$

satisfying

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I} \vdash u : \mathsf{inh}\ A}{\Gamma \vdash (\mathsf{squeeze}\ u)(i0) = \mathsf{transp}\ u(i0) : \mathsf{inh}\ A(i1)} \qquad \frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I} \vdash u : \mathsf{inh}\ A}{\Gamma \vdash (\mathsf{squeeze}\ u)(i1) = u(i1) : \mathsf{inh}\ A(i1)}$$

by the equations

$$\begin{array}{lcl}
\mathsf{transp}\ (\mathsf{inc}\ a) & = & \mathsf{inc}\ (\mathsf{comp}^i\ A\ []\ a) \\
\mathsf{transp}\ (\mathsf{squash}(u_0, u_1, r)) & = & \mathsf{squash}(\mathsf{transp}\ u_0, \mathsf{transp}\ u_1, r) \\
\mathsf{transp}\ (\mathsf{hcomp}^j\ [\varphi \mapsto u]\ u_0) & = & \mathsf{hcomp}^j\ [\varphi \mapsto \mathsf{transp}\ u]\ (\mathsf{transp}\ u_0) \\
 & & \\
\mathsf{squeeze}\ (\mathsf{inc}\ a) & = & \mathsf{inc}\ (\mathsf{comp}^j\ A(i \vee j)\ [(i = 1) \mapsto a(i1)]\ a) \\
\mathsf{squeeze}\ (\mathsf{squash}(u_0, u_1, r)) & = & \mathsf{squash}(\mathsf{squeeze}\ u_0, \mathsf{squeeze}\ u_1, r)
\end{array}$$

and we define $\mathsf{squeeze}\ (\mathsf{hcomp}^j\ [\delta \mapsto u, \varphi_0 \wedge (i = 0) \mapsto u_0, \varphi_1 \wedge (i = 1) \mapsto u_1]\ v)$ as

$$\mathsf{hcomp}^j\ [\delta \mapsto \mathsf{squeeze}\ u, \varphi_0 \wedge (i = 0) \mapsto \mathsf{transp}\ u_0, \varphi_1 \wedge (i = 1) \mapsto u_1]\ (\mathsf{squeeze}\ v)$$

using the fact that any formula φ has a decomposition $\delta \vee (\varphi_0 \wedge (i = 0)) \vee (\varphi_1 \wedge (i = 1))$ where δ is the disjunction of all faces of φ not containing i, and φ_0 (resp. φ_1) the disjunction of all faces α such that $\alpha \wedge (i = 0)$ (resp. $\alpha \wedge (i = 1)$) is a face of φ.

Using these operations, we can define

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{inh}\ A \qquad \Gamma \vdash u_0 : \mathsf{inh}\ A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{inh}\ A(i1)[\varphi \mapsto u(i1)]}$$

by $\Gamma \vdash \mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 = \mathsf{hcomp}^i\ [\varphi \mapsto \mathsf{squeeze}\ u]\ (\mathsf{transp}\ u_0) : \mathsf{inh}\ A(i1)$ if $\varphi \neq 1$ and $\Gamma \vdash \mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 = u(i1) : \mathsf{inh}\ A(i1)$ if $\varphi = 1$.

Given $\Gamma \vdash B$ and $\Gamma \vdash q : (x\ y : B) \to \mathsf{Path}\ B\ x\ y$ and $f : A \to B$ we define $g : \mathsf{inh}\ A \to B$ by the equations

$$\begin{array}{lcl}
g\ (\mathsf{inc}\ a) & = & f\ a \\
g\ (\mathsf{squash}(u_0, u_1, r)) & = & q\ (g\ u_0)\ (g\ u_1)\ r \\
g\ (\mathsf{hcomp}^j\ [\varphi \mapsto u]\ u_0) & = & \mathsf{comp}^j\ B\ [\varphi \mapsto g\ u]\ (g\ u_0)
\end{array}$$

Appendix 4: How to build a path from an equivalence

Given $\Gamma \vdash \sigma : \mathsf{Equiv}(A, B)$ we define

$$\Gamma, i : \mathbb{I} \vdash E = \mathsf{glue}(B, [(i = 0) \mapsto \sigma, (i = 1) \mapsto \mathsf{id}_B])$$

where $\mathsf{id}_B : \mathsf{Equiv}(B, B)$ is defined as

$$\mathsf{id}_B = (\lambda x : B.x, \lambda x : B.((x, 1_x), \lambda u : (y : B, \mathsf{Path}\ B\ x\ y).\langle i\rangle(u.2\ i, \langle j\rangle u.2\ (i \wedge j))))$$

We have then $\Gamma, i : \mathbb{I}, (i = 0) \vdash E = A$ and $\Gamma, i : \mathbb{I}, (i = 1) \vdash E = B$, so that $E(i0) = A$ and $E(i1) = B$.

If we now introduce an universe U by reflecting all typing rules and

$$\frac{}{\Gamma \vdash U} \qquad \frac{\Gamma \vdash A : U}{\Gamma \vdash A}$$

we can define $\mathsf{Equiv}(A, B) \to \mathsf{Path}\ U\ A\ B$ by $\lambda u : \mathsf{Equiv}(A, B).\langle i\rangle \mathsf{glue}(B, [(i = 0) \mapsto \sigma, (i = 1) \mapsto \mathsf{id}_B])$.

Appendix 5: Semantics

Let $\mathcal{C}$ the following category. The objects are finite sets $I, J, \ldots$. A morphism $\mathsf{Hom}(J, I)$ is a map $I \to \mathsf{dM}(J)$ where $\mathsf{dM}(J)$ is the free de Morgan algebra on J. The presheaf $\mathbb{I}$ is defined by $\mathbb{I}(J) = \mathsf{dM}(J)$. The presheaf $\mathbb{F}$ is defined by taking $\mathbb{F}(J)$ to be the free distributive lattice generated by formal elements $(j = 0)$, $(j = 1)$ for j in J, with the relations $(j = 0) \wedge (j = 1) = 0$.

We interpret context as presheaves over the category $\mathcal{C}$. A dependent type $\Gamma \vdash A$, non necessarily "fibrant", is interpreted as a family of sets $A\rho$ for each I and $\rho \in \Gamma(I)$ together with restriction maps $A\rho \to A\rho f$, $u \longmapsto uf$ for $f : J \to I$, satisfying $u1_I = u$ and $(uf)g = u(fg) \in \Gamma(K)$ if $g : K \to J$. An element $\Gamma \vdash a : A$ is interpreted by a family $a\rho \in A\rho$ for I and $\rho \in \Gamma(I)$, such that $(a\rho)f = a(\rho f) \in A\rho f$ if $f : J \to I$.

If $\Gamma \vdash A$, we interpret $\Gamma.A$ as the cubical set defined by taking $(\Gamma.A)(I)$ to be the set of element ρ, u such that $\rho \in \Gamma(I)$ and $u \in A\rho$. If $f : J \to I$ the restriction map is defined by $(\rho, u)f = \rho f, uf$.

If $\Gamma.A \vdash B$ and $\Gamma \vdash a : A$ we define $\Gamma \vdash B[a]$ by taking $B[a]\rho$ to be the set $B(\rho, a\rho)$.

If $\Gamma \vdash \varphi : \mathbb{F}$ then $\varphi\rho \in \mathbb{F}(I)$ for each $\rho \in \Gamma(I)$. We define $(\Gamma, \varphi)(I)$ to be the set $\rho \in \Gamma(I)$ such that $\varphi\rho = 1$. (In particular $(\Gamma, 0)(I)$ is empty.)

If $\Gamma \vdash A$ and ρ is in $\Gamma(I)$ and φ is in $\mathbb{F}(I)$, we define a *partial element of $A\rho$ of extent φ* to be a family of elements u_f in $A\rho f$ for $f : J \to I$ such that $\varphi f = 1$, satisfying $u_f g = u_{fg}$ if $g : K \to J$.

We define next when $\Gamma \vdash A$ has a *composition structure*. This is given by a family of operations $\mathsf{comp}^i\ A\rho\ [\varphi \mapsto u]\ a_0$ in for ρ in $\Gamma(I, i)$, φ in $\mathbb{F}(I)$ and u a partial element of $A\rho$ of extent φ and a_0 in $A\rho(i0)$ such that $a_0 f = u_f(i0)$ if $\varphi f = 1$. This element should satisfy $(\mathsf{comp}^i\ A\rho\ [\varphi \mapsto u]\ a_0)f = u_f(i1)$ if $\varphi f = 1$. Furthermore, we have the *uniformity condition*

$$(\mathsf{comp}^i\ A\rho\ [\varphi \mapsto u]\ a_0)g = \mathsf{comp}^j\ (A\rho(g, i = j))\ [\varphi g \mapsto u(g, i = j)]\ a_0 g$$

if $g : J \to I$ and j not in J.

It is then possible to give the semantics of the composition operations. If $\Gamma.\mathbb{I} \vdash A$ and $\Gamma \vdash \varphi$ and $\Gamma.\mathbb{I}, \varphi\mathsf{p} \vdash u : A$ and $\Gamma \vdash a_0 : A[0][\varphi \mapsto u[0]]$ and ρ is in $\Gamma(J)$ we define

$$(\mathsf{comp}\ A\ [\varphi \mapsto u]\ a_0)\rho = \mathsf{comp}^j\ A(\rho, j)\ [\varphi\rho \mapsto u(\rho, j)]\ a_0\rho$$

for j not in J.

Appendix 6: Universes have a composition operation

Given $\Gamma \vdash A$, $\Gamma \vdash B$ and $\Gamma, i : \mathbb{I} \vdash E$ such that $E(i0) = A$ and $E(i1) = B$ we explain first how to buid $\Gamma \vdash \mathsf{equiv}^i\ E : \mathsf{Equiv}(A, B)$.

We define

$$\Gamma \vdash f : A \to B \qquad \Gamma \vdash g : B \to A \qquad \Gamma, i : \mathbb{I} \vdash u : A \to E \qquad \Gamma, i : \mathbb{I} \vdash v : B \to E$$

such that $u(i1) = f$ and $u(i0) = \lambda x : A.x$ and $v(i0) = g$ and $v(i1) = \lambda y : B.y$. The definitions are

$$\begin{array}{rcl} f & = & \lambda x : A.\mathsf{comp}^i\ E\ []\ x \\ g & = & \lambda y : B.\mathsf{comp}^i\ E(1-i)\ []\ y \\ u & = & \lambda x : A.\mathsf{fill}^i\ E\ []\ x \\ v & = & \lambda y : B.\mathsf{fill}^i\ E(1-i)\ []\ y \end{array}$$

We then show that two elements (x_0, β_0) and (x_1, β_1) in $(x : A, \mathsf{Path}\ B\ y\ (f\ x))$ are path-connected. This is obtained by the definitions

$$\begin{array}{rcl} \omega_0 & = & \mathsf{comp}^i\ E(1-i)\ [(j=0) \mapsto v\ y, (j=1) \mapsto u\ x_0]\ (\beta_0\ j) \\ \omega_1 & = & \mathsf{comp}^i\ E(1-i)\ [(j=0) \mapsto v\ y, (j=1) \mapsto u\ x_1]\ (\beta_1\ j) \\ \theta_0 & = & \mathsf{fill}^i\ E(1-i)\ [(j=0) \mapsto v\ y, (j=1) \mapsto u\ x_0]\ (\beta_0\ j) \\ \theta_1 & = & \mathsf{fill}^i\ E(1-i)\ [(j=0) \mapsto v\ y, (j=1) \mapsto u\ x_1]\ (\beta_1\ j) \\ \omega & = & \mathsf{comp}^j\ A\ [(k=0) \mapsto \omega_0, (k=1) \mapsto \omega_1]\ (g\ y) \\ \theta & = & \mathsf{fill}^j\ A\ [(k=0) \mapsto \omega_0, (k=1) \mapsto \omega_1]\ (g\ y) \end{array}$$

so that we have $\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \theta_0 : E$ and $\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \theta_1 : E$ and $\Gamma, j : \mathbb{I}, k : \mathbb{I} \vdash \theta : A$. If we define

$$\delta = \mathsf{comp}^i\ E\ [(j=0) \mapsto v\ y, (j=1) \mapsto u\ \alpha, (k=0) \mapsto \theta_0, (k=1) \mapsto \theta_1]\ \theta$$

we then have $\langle k\rangle(\alpha, \langle j\rangle\theta) : \mathsf{Path}\ (x : A, \mathsf{Path}\ B\ y\ (f\ x))\ (x_0, \beta_0)\ (x_1, \beta_1)$ as desired.

Since $(x : A, \mathsf{Path}\ B\ y\ (f\ x))$ is inhabited, since it contains the element $(g\ y, \gamma)$ where $\gamma = \langle k\rangle\mathsf{comp}^i\ E\ [(k=0) \mapsto v\ y, (k=1) \mapsto u\ (g\ y)]\ (g\ y)$, we have shown that the fiber of f at y is contractible. Hence f is an equivalence and we have built $\mathsf{equiv}^i\ E : \mathsf{Equiv}(A, B)$.

If we now introduce an universe U by reflecting all typing rules and

$$\frac{}{\Gamma \vdash U} \qquad \frac{\Gamma \vdash A : U}{\Gamma \vdash A}$$

then we can define $\mathsf{comp}^i\ U\ [\varphi \mapsto E]\ A_0 = \mathsf{glue}(A_0, \varphi \mapsto (E(i1), \mathsf{equiv}^i\ E(1-i)))$.

Appendix 7: Univalence

We have shown how to build maps $\mathsf{Path}\ U\ A\ B \to \mathsf{Equiv}(A, B)$ and $\mathsf{Equiv}(A, B) \to \mathsf{Path}\ U\ A\ B$. Using only the glueing operation, it has been shown formally by Simon Huber and Anders Mörtberg that these two maps are homotopy inverse.

Since one can prove formally that a map with a homotopy inverse is an equivalence and that the map $\mathsf{Path}\ U\ A\ B \to \mathsf{Equiv}(A, B)$ is equal to the one we get by path elimination and the canonical proof of $\mathsf{Equiv}(A, A)$, we get univalence for Path.

It can then be shown formally that univalence for $\mathsf{Id}\ U\ A\ B$ holds as well.

Another approach is to show that the type $(X : U, \mathsf{Equiv}(X, A))$ is contractible. (This is one possible way to state the univalence axiom.) For this it is enough to show that any partial element of this type $\varphi \vdash (T, \sigma)$ can be extended to a total element. And for this, it is enough to show that the map $\mathsf{unglue} : B \to A$, where $B = \mathsf{glue}([\varphi \mapsto (T, \sigma)], A)$ is an equivalence.

For showing this, we give $\psi \vdash b = \mathsf{glue}([\varphi \mapsto b], a) : B$ and $u : A[\psi \mapsto a]$ and we explain how to build

$$\tilde{b} : B[\psi \mapsto b] \qquad \alpha : \mathsf{Path}\ A\ u\ (\mathsf{unglue}\ \tilde{b})[\psi \mapsto \langle i\rangle u]$$

Since $\varphi \vdash \sigma : T \to A$ is an equivalence and we have $\psi, \varphi \vdash b : T$ and $\psi, \varphi \mapsto \sigma\ b = a : A$ we can find $\varphi \mapsto t : T[\psi \mapsto b]$ and $\varphi \vdash \beta : \mathsf{Path}\ A\ u\ (\sigma t)[\psi \mapsto \langle i\rangle u$. We then define $\tilde{a} = \mathsf{comp}^i\ A\ [\varphi \mapsto \beta\ i, \psi \mapsto u]\ u$ and $\alpha = \mathsf{fill}^i\ A\ [\varphi \mapsto \beta\ i, \psi \mapsto u]\ u$. We then conclude by taking $\tilde{b} = \mathsf{glue}([\varphi \mapsto t], \tilde{a})$.