A 4-hours Course in Recursive Game Theory for Classical Arithmetic

An effective interpretation of classical proofs as trial and error algorithms through the game theoretical notion of backtracking

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Talk 1. Coquand's Semantics of Evidence

If we add to Tarski games the possibility of retracting each move finitely many times, we obtain an effective, sound and complete model of Truth for Classical Arithmetic

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A <<**Semantics of Evidence**>> for Peano Arithmetic PA

- In an unpublished work, rediscovered in 1970 [Bern], Gentzen provided a proof that PA is consistent, a proof which may be read as a game theoretical interpretation of PA.
- In June 1991, T. Coquand [Coq91] built over this work, providing an interpretation of proofs of PA as **recursive** winning strategies of a suitable game.
- Coquand calls his work a **Semantics of Evidence**, because it interprets a classical proofs as the **construction of an evidence** for its thesis.

Tarski Games for Arithmetical Formulas

- In a Tarski game, the play is between **Nature** (also called **Abelard**, the opponent) and **Myself** (also called **Eloise**) Myself tries to defend the truth of the formula while Nature attack it.
- We first give a semantics for **closed prenex** arithmetical formulas only: A = ∀x∃y∀z … p(x,y,z,…) , for any **primitive recursive** p.
- **Rules of the play.** When the formula is existential, Myself chooses an instance of it, when the formula is universal, Nature chooses an instance of it.
- When we reach an atomic instance p(a,b,c,…), if it is true then Myself wins, if it is false Nature wins.

Tarski Games in Set Theory (with non-recursive strategies)

- Myself has a (usually non-recursive) winning strategy for A if and only if **A is true**.
- Nature has a (usually non-recursive) winning strategy for A if and only if **A is false**.
- The optimal strategy for Myself is to choose **some true instance** of A, if any, and to move 0 otherwise.
- The optimal strategy for Nature is to choose **some false instance** of A, if any, and to move 0 otherwise.
- The objection to this semantics is: we only **recopy the classical definition of truth** in game theoretical terms.

Tarski Games restricted to recursive strategies

- If we want an effective semantics, a proof should be interpreted by a **recursive winning strategy**.
- However, for most theorems of PA, say A =

∀x∃y∀z. q(x,y,z) (with q(x,y,z)⇔p(x,y)∨~p(x,z))

- there is no recursive winning strategy (see next slide). Thus, Tarski Games are no effective interpretation of PA.
- The problem is that we impose **no limitation whatsoever** to the way Nature selects its moves, and therefore Nature is **much stronger** than Myself.

Proof: there is no recursive winning strategy for ∀x∃y∀z.q(x,y,z)

- Choose any **non-recursive** Σ^0_1-predicate ∃y.p(x,y), and define q(x,y,z) ⇔ p(x,y)∨~p(x,z).
- Assume there is a recursive map b=ϕ(a), selecting our move y=b out of the move x=a of Nature, in such a way to win the game for ∀x∃y∀z.q(x,y,z).
- Then for all x, z the formula q(x,ϕ(x),z), that is, p(x,ϕ(x))∨~p(x,z), is true. If p(x,ϕ(x)) is true, then ∃y.p(x,y) is true. If p(x,ϕ(x)) is false, then ~p(x,z) is true for all z, therefore ∃y.p(x,y) is false.
- Thus, ∃y.p(x,y), being equivalent to p(x,ϕ(x)), is a **recursive** predicate. **Contradiction**.

Retracting a previous move

- In order to counteract the overwhelming power of Nature, we allow Myself to **retract any of his previous moves finitely many times**, and to move again from this position, restarting the game.
- However, if the play is **infinite** then Myself **loses**.
- This bonus to Myself produces a **perfect balance of power**: we may classically prove that there is a winning strategy for Myself over a prenex formula of PA if and only if the formula is true (see later).
- Coquand called the retraction of a previous move:

Backtracking

There a recursive winning strategy for $\forall x \exists y \forall z.q(x,y,z)$ **using backtracking**

- Nature plays x=a. Myself answer y=0.
- Nature plays z=c.
- If q(a,0,c), that is, $p(a,0) \vee \sim p(a,c)$, then Myself wins.
- Otherwise **p(a,c) is true**. Myself **retracts his first move y=0**. This times, Myself **moves y=c**.
- Nature plays any z=d. Myself wins, because the final position is $p(a,c) \vee \sim p(a,d)$, and p(a,c) is true.
- Myself wins with a strategy by trial and error. Myself **uses his mistakes to improve** his moves.

There a recursive winning strategy for $\exists x \forall y.f(x) \leq f(y)$ (with $f:Nat \rightarrow Nat$) **using backtracking**

- Myself plays $x=a_0=0$.
- Nature plays $y=a_1$ such that $f(a_0)>f(a_1)$
- **Myself retracts his first move $x=a_0$**. This times, Myself **moves $x=a_1$**.
- Nature plays $y=a_2$ such that $f(a_1)>f(a_2)$.
- **Myself retracts his second move $x=a_1$**. This times, Myself **moves $x=a_2$**.
- Since $f(a_0)>f(a_1)>f(a_2)>f(a_3)>\ldots$, eventually Nature plays some $y=a_{i+1}$ such that $f(a_i) \leq f(a_{i+1})$ and loses.

Generalization to all **closed formulas** of PA

- We call $\vee$, $\exists$, $\wedge$, $\forall$ the **positive** connectives: $\vee$, $\exists$ are **disjunctive** and $\wedge$, $\forall$ **conjunctive**.
- **Rules for $\vee$, $\wedge$.** When the formula is **$A \vee B$**, Myself chooses either A or B. When the formula is **$A \wedge B$**, Nature chooses either A or B.
- We define $A^{\perp}$ as the **dual of A**, by switching all $\vee$ with $\wedge$, all $\exists$ with $\forall$, and each closed atomic formula p(a,b,c,…) with $p^{\perp}$(a,b,c,…), with $p^{\perp}$ the primitive recursive predicate complement of p.
- In the original Coquand's formalization there are no connective **$A \Rightarrow B$** and **$\sim A$**: they are represented by $A^{\perp} \vee B$ and $A^{\perp}$.

Closed sequents of PA interpreting plays

- A sequent $\Gamma = A_1, \ldots, A_n$ interprets a (possibly unfinished) play with backtracking if all formulas are **closed**, all but at most the last one are **disjunctive**, and:

1. A_n is the current position.
2. the formulas A_i with i<n are all previous positions of the play from which Myself moved, in the same order they have in the play.

- A sequent Γ is **valid** if there is some recursive winning strategy from the position of the play interpreted by Γ.

Closed sequents of PA interpreting plays

Let **A=∀x∃y∀z.q(x,y,z)** (q(x,y,z) ⇔ p(x,y)∨~p(x,z))

- Γ ={A} interprets the initial position of the play.
- Γ ={∃y∀z.q(a,y,z)} interprets the play after Nature moves x=a.
- Γ ={∃y∀z.q(a,y,z), ∀z.q(a,0,z)} interprets the play after Myself moves y=0.
- Γ ={∃y∀z.q(a,y,z), q(a,0,c)} interprets the play after Nature moves z=c.
- Γ ={∃y∀z.q(a,y,z), q(a,0,c), ∀z.q(a,c,z)} interprets the play after Myself retracts his first move y=0 and moves y=c this time.

Interpreting some **closed sequents** of PA

- Γ ={∃y∀z.q(a,y,z), q(a,0,c), ∀z.q(a,c,z), q(a,c,d)} interprets the play after Nature moves z=d.
- This is the final position of the play. Since q(x,y,z) ⇔ p(x,y)∨~p(x,z), then, as we already explained:

1. either q(a,0,c) is true, and the play terminates with q(a,0,c),
2. or q(a,c,d) is true, no matter what is the value of d, and the play terminates with q(a,c,d).

In which sense we have a Semantics of Evidence?

- The note of June, 1991 is quite informal but raises two precise problems about the semantics of evidence.
- **Problem 1.** How much **are the moves of a winning strategy for Myself reliable**?
- In general, the moves of Myself are only **attempts** to find the right move.

The moves of a winning strategy for Myself are reliable for Π^0_2-formulas

- Indeed, a winning strategy for a formula $\forall x \exists y.p(x,y)$, with p symbol for primitive recursive predicate, must effectively provide for each x=a some y=b such that p(a,b) is true.
- Instead, Myself may win a game for $\exists x \forall y.p(x,y)$ by playing a **false** witness x=a for $\exists y \forall z.(p(x,y)$, provided Nature is unable to find some y=b such that ~p(a,b), even if **such b exists**.
- If A is not a Π^0_2-formula, a winning strategy may produce a **wrong** witness.

The interest of Π^0_2-formulas

- **Π^0_2-formula are** specifications of algorithms and play a **central** role in an applicative view of Mathematics.
- Formulas which are **not Π^0_2-formulas** may be considered as **Lemmas** in the derivation of a Π^0_2-formula. Even if win the play we are not sure we have a correct witness for them.
- However, even an **imprecise evidence** for a Lemma which is not Π^0_2 may be enough for producing **a precise evidence** for a Π^0_2-thesis.

In which sense we have a Semantics of Evidence?

- **Problem 2.** Are games with backtracking **a sound interpretation for all logical rules?** That is: if all premises of a rule of Sequent Calculus are valid, is the conclusion valid?
- Modus ponens and Cut rule are problematic. Given a recursive winning strategy **for A and for $A \Rightarrow B$** = $A^\perp \vee B$ we should produce a winning strategy **for B**.
- This is easy to do for Tarski games, but **problematic** in the case of games with backtracking.

The symmetry of Tarski Games makes the interpretation of cut easy

- Let A, B be any formula of PA.
- Any strategy for Myself in a Tarski game over A corresponds to a strategy for Nature in a Tarski game over $A^\perp$, and conversely. The reason is that a strategy for Myself moves over disjunctive subformulas of A, which correspond to conjunctive subformulas of $A^\perp$, those on which Nature moves.
- Therefore we may interpret Modus Ponens over Tarski games by playing the winning strategy for $A^\perp \vee B$ against the winning strategy for A over $A^\perp$, and against Nature over B.

The asymmetry of the Semantics of Evidence makes Cut rule problematic

- Let A be any formula of PA.
- Most strategies for Myself in a game with backtracking over A do not correspond to a strategy for Nature in a game with backtracking over $A^{\perp}$, because Myself may backtrack, while Nature cannot.
- The solution proposed by Coquand it to allow both players to backtrack, but **to reduce the visibility of past moves for each of them**, in such a way that each player **does not see that the other player backtracks.**

An asymmetric view of past moves restores the asymmetry of cut rule

- Whenever a player comes back to a previous move and changes it, he **makes invisible** to the other player **all moves from the move he changes**. The other player does not perceive the backtracking done by his opponent.
- For each player in a cut rule, the **other player looks like Nature**, non-recursive and non-deterministic, while in fact he looks so only for lack of information.
- This **symmetric illusion** restore the asymmetry Myself/Nature in the interpretation of cut rule.

Toward a formal definition of game with backtracking and cut

- It not easy to provide a mathematical description of the symmetric illusion making each player non-recursive, non-deterministic and without backtracking to the eyes of the other player.
- T. Coquand solves this problem in a message to the type net of Jan., 4 1992, introducing the combinatorial notions of Interaction Sequence and view.

Interaction Sequences

- An interaction sequence is any map $f:\{1,\dots,n\}\rightarrow\{0,\dots,n-1\}$, such that $f(i)<i$ and i, $f(i)$ have the same parity.
- The integers are indexes of moves in a game with backtracking. The player doing the move of index i moves back to f(i), erases all moves after f(i) if any from the memory of the other player, then moves.
- Coquand assumes that **players alternate**. One player moves at even positions and comes back to even positions. The other player moves at odd positions and comes back to odd positions.

A notion of visibility

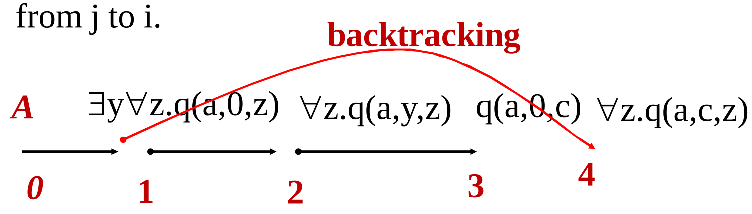
- There is a notion of **visibility for each player**. If a player makes a move invisible to the other, we say that he **makes the other to forget the move**.

1. The move number j is immediately **visible from player p from move i** if either p does the move i and j=i-1, or the opponent of p does the move i and j=f(i).
2. Visibility is the **transitive closure** of immediate visibility.
3. If p does the moves i we require that **f(i) is visible by p from i-1.**

- **f(i)** is the move to which the opponent of p **comes back before moving i**.

An example of visibility

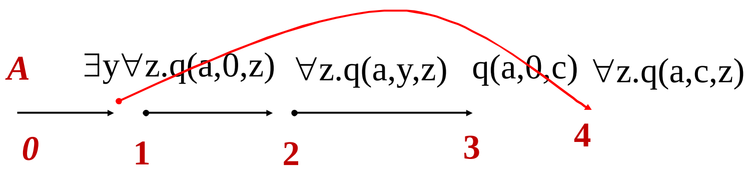
- We introduce the sequences of interactions for the play from **A=∀x∃y∀z.q(x,y,z)** (with q(x,y,z) ⇔ p(x,y)∨~p(x,z)). We represent j=f(i) by an arrow from j to i.



We have f(0)=1, f(2)=1, f(3)=2, f(4)=1. In the move 4, Myself comes back to ∃y∀z.q(a,0,z), then retracts the move ∀z.q(a,y,z) and moves ∀z.q(a,c,z).

An example of visibility

Myself always sees **all** previous moves:



Nature cannot see the moves in]1,4[, because Myself erased them from its view, when he retracted the move 2: ∀z.q(a,y,z) and did the move 4:∀z.q(a,c,z)

The formal definition of a game with backtracking and cut rule

- We recall that moves in a play with backtracking and cut have an additional visibility constraint:

if p does the move i, then f(i) is visible by p from i

- This clause concludes the **definition of game with backtracking and cut**.
- This clauses expresses the fact that the opponent of p forced p to forget the existence of some moves, and p cannot backtrack to a move of which he ignores the existence.

Lifting of a strategy to a game with cut

- If we have a recursive strategy for Myself in a game with backtracking but without cut (in which Nature cannot backtrack), we may lift it to recursive strategy in a a game with cut (in which Nature backtracks).
- The reason is that the part of a play visible by Myself is always cut-free: Myself cannot catch Nature to backtrack, becomes when this happen, Nature makes Myself to forget that this happened.
- We call these strategies **lifted strategies**.

Cut-elimination theorem for game theory

Theorem ([Coq95] 1995). The cut between a lifted recursive winning strategy over A and a lifted recursive winning strategy over $A^{\perp} \vee B$ is a lifted winning strategy over B.

Proof. We give a hint in the next talk.

- To put otherwise: Cut rule preserves the validity of a sequent in the game interpretation with backtracking. Cut rule is interpreted by a **dialogue** between strategies about **the truth of the cut formula**.

The visibility makes the interpretation of Cut rule possible

- Let A, B be any formula of PA.
- Any strategy for Myself in a game with backtracking and cut over A corresponds to a strategy for Nature over $A^{\perp}$, and conversely, because now **both players may backtrack**. The strategy for Myself moves over **disjunctive subformulas** of A, which correspond to **conjunctive subformulas** of $A^{\perp}$, those on which Nature moves.
- Therefore we may interpret Modus Ponens over Tarski games by playing the winning strategy for $A^{\perp} \vee B$ **against the winning strategy for A** over $A^{\perp}$, and **against Nature over B**.

Talk 2. The Cut-Elimination and the Isomorphism Theorems for Games with Backtracking

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Abstract of the talk

- We give some proof hint of the Cut-Elimination Theorem for Games with backtracking: the Cut rule preserve the validity of a formula in the Game semantics.
- We include another result of interest: arithmetical proof of a formula A of PA_ω (Peano Arithmetic extended with recursive ω-rule) are in one-to-one correspondence with recursive winning strategies for Eloise on the game with backtracking for A.
- Theorems of PA_ω are exactly the classically true arithmetical formula: thus, games with backtracking are a recursive, sound and complete interpretation of truth for classical arithmetic.

The language of PA

- The formulas of **PA** are all first order arithmetical formulas we may define from symbols for all primitive recursive functions and predicates with $\forall,\wedge,\vee,\exists$.
- A closed atomic formula is conjunctive if it is true and disjunctive if it is false. A formula with first symbol $\forall,\wedge$ is conjunctive, with first symbol $\vee,\exists$ is disjunctive.
- We write $\mathbf{A} = \vee_{i\in I}\mathbf{A}_i$, $\wedge_{i\in I}\mathbf{A}_i$ for a generic conjunctive, disjunctive formula of immediate subformulas $\{\mathbf{A}_i|_{i\in I}\}$: $I=\emptyset$ if A atomic, $I=\{1,2\}$ if $A=B\vee C, B\wedge C$, and $I=\{\text{closed terms}\}$ if $A= \forall x.B$, $\exists x.B$.

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Formulas as games

- Any formula F of PA is interpreted as a game with backtracking.
- The moves of the game for F are the nodes of the subformula tree for F.
- E (Myself or Eloise) moves from a disjunctive subformula. A (Nature or Abelard) moves from a conjunctive subformula.
- For sake of simplicity we assume as in [Coq95] that label are alternating in every branch (we could bypass this extra hypothesis).

Games with backtracking

- A partial play with backtracking and cut in F is a pair $<m_0, m_1,\ldots,m_n>$ of a list of moves (nodes of the subformula tree of F) and a map $f:\{1,\ldots,n\}\rightarrow\{0,\ldots,n-1\}$ such that for all $i\in \{1,\ldots,n\}$:

1. $m_0 = F$, $m_{f(i)}$ is the father node of m_i.
2. $f(i)<i$ and $i,f(i)$ have a different parity.
3. $f(i)$ is visible from $i-1$ by the player moving from m_{i-1}.
4. Visibility of player p is the sequence $i-1$, $f(i-1)$, $f(i-1)-1$, $f(f(i-1)-1)-1$, … if p has the parity of $i-1$, it is the sequence $f(i)$, $f(i)-1$, $f(f(i)-1)$, … otherwise.

Strategies

- We order plays $<L,f> \leq <L',f'>$ if the list L of moves is a prefix of L' and the map $f:\{1,\ldots,n\}\rightarrow\{0,\ldots,n-1\}$ is a restriction of $f':\{1,\ldots,n'\}\rightarrow\{0,\ldots,n'-1\}$.

Def. A **(lifted) strategy** σ for player p in F is a tree of plays in F w.r.t. the ordering $\leq$, such that:

1. if p moves from $<L,f>\in\sigma$ then there is at most one successor $<L_1,f_1>$ of $<L,f>$ in σ,
2. whenever the opponent of p moves from $<L,f>\in\sigma$, then all successors of $<L,f>$ are in σ.
3. if the p-visible parts of $<L,f>$, $<L',f'>\in\sigma$ are the same and $<L_1,f_1>$, $<L'_1,f'_1>$ are their successors in σ, then the p-visible parts of $<L_1,f_1>$, $<L'_1,f'_1>$ are the same

Winning Strategies and dialogues

- A strategy for p is **terminating** if the p-visible parts of all $<L,f>\in\sigma$ form a **well-founded tree**.
- A strategy σ for p is **winning** if σ is terminating and whenever p moves from $<L,f>\in\sigma$ then there is **exactly one successor** of $<L,f>$ in σ.
- Assume σ, τ are strategies for E, A on a formula F. We call the **dialogue** between σ, τ and we denote with $\sigma*\tau$ the maximal play between σ, τ.
- $\sigma*\tau$ is unique because each strategy has at most one move from each position. $\sigma*\tau$ may be infinite.

The cut operator

- Given a strategy σ of E on $F\vee G$ and a strategy τ of E on $F^{\perp}$ we may define a strategy $Cut(\sigma, \tau)$ of E on G.
- Indeed, from τ we may define a strategy $\tau^{\perp}$ of A on F, by switching E and A. In $Cut(\sigma, \tau)$, E follows σ, and plays against A on G, and against the strategy $\tau^{\perp}$ of E on F.
- We define $Cut(\sigma, \tau)$ by skipping all moves on $F\vee G$, F.
- $Cut(\sigma, \tau)$ may fail to answer to a move of A on G because the play on F against $\tau^{\perp}$ may be infinite.
- However, uniqueness of the move on G is preserved by Cut, therefore if σ, τ are strategies then $Cut(\sigma, \tau)$ is.
- Given $F = \vee_{i<r} F_i$ and for all $j\in X\subseteq[0,r[$ some strategy τ_j for A on F_j, we may define $Cut(\sigma, \{\tau_j | j\in X\})$.

The Cut-elimination Theorem for Games with backtracking

- Our aim is to prove: given a winning strategy σ of E on $F \vee G$ and a winning strategy τ of E on $F^{\perp}$ the strategy $Cut(\sigma, \tau)$ of E on G is winning.
- We prove first the following Lemma. Assume $F = \vee_{i<r} F_i$ and σ is a terminating strategy for E on F.

1. If τ is a terminating strategy for A on F then $\sigma * \tau$ is finite.
2. If for some i,r and all $j \neq i$ we have a terminating strategy τ_j for A on F_j, then $Cut(\sigma, \{\tau_j | j \neq i\})$ is a terminating strategy for E on F_i, and every finite play on F_i is obtained from some finite play on F.

Proof of the Main Lemma

- We argue by principal induction $F = \vee_{i<r} F_i$ and secondary induction over the ordinal height of the strategy σ for E on F.

1. Let τ be a terminating strategy for A on F. We have to prove that the dialogue $\sigma * \tau$ is finite.

- Assume that the first move of σ is some F_i. From now on, σ may move on the subformula F_i, or may backtrack to F and choose some j<r and move from F_j. Thus, σ defines a strategy σ' for E on $F' = F_i \vee F_0 \vee F_1 \vee F_2 \vee \ldots$, F' has the same height as F, while σ' has ordinal height less than σ.

Proof of point 1 of the Main Lemma

- Let j<r. The terminating strategy τ for A on F_j defines a terminating strategy τ_j for A on F_j: τ_j is obtained by skipping the initial position F from the set of all plays in τ in which E first moves F_j, then never backtracks to F.

1. The part of the play on F_i may be described as the dialogue $Cut(\sigma', \{\tau_j | j<r\}) * \tau_i$. By secondary induction hypothesis on σ', the strategy $Cut(\sigma', \{\tau_j | j<r\})$ is terminating. Thus, the dialogue $Cut(\sigma', \{\tau_j | j<r\}) * \tau_I$ is terminating by principal induction hypothesis on F_i, and it is obtained from a finite play of σ', hence of σ. less than σ.

Proof of point 2 of the Main Lemma

2. Assume that for some i,r and all $j \neq i$ we have a terminating strategy τ_j for A on F_j. We have to prove that $Cut(\sigma, \{\tau_j | j \neq i\})$ is a terminating strategy for E on F_i, and every finite play on F_i is obtained from some finite play on F.

- Between any two moves on G, the strategy $Cut(\sigma, \tau)$ may play against the winning strategy $\tau^{\perp}$ for A on $F^{\perp}$. This dialogue is finite by point 1.
- Thus, we may map any visible part of a play $<L,f> \in Cut(\sigma, \{\tau_j | j \neq i\})$ into the visible part of some play $<L',f'>$ of σ. From σ termining we conclude the thesis.

The Cut-elimination Theorem for Games with backtracking

- Given a winning strategy σ of E on F∨G and a winning strategy τ of E on $F^\perp$ the strategy Cut(σ, τ) of E on G is winning.
- Proof. By the Main Lemma, point 2, Cut(σ,τ) is terminating, and every finite play is obtained from a finite play of σ. Assume E moves from <L,f>∈Cut(σ,τ). If there is no successor of <L,f> in Cut(σ, τ), then <L',f'> is obtained from a terminated play in σ, in which E wins on $F^\perp$. But this is impossible because τ is a winning strategy for A on on $F^\perp$.

A formulation of PA_ω satisfying the proof/strategy isomorphism

- The formulas of **PA_ω** are the formula of **PA**.
- Sequents of **PA_ω** are **ordered lists** of closed formulas.
- Contraction and Exchange rules are **not built-in** in the notion of sequent.
- We hyde Exchange rule through the fact that the active formula, if disjunctive, may be in any position in the sequent.
- Identity rule is trivially derivable in **PA_ω**.
- **Cut rule is derivable but not trivial at all.**

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A formulation of PA_ω with 2 rules
(in one-side form, with judgements)

$$\frac{\Gamma, \vee_{i\in I}A_i, \Delta, A_i}{\Gamma, \vee_{i\in I}A_i, \Delta}$$ ***(disj. with implicit contraction and exchange: for some i∈I)***

$$\frac{\Gamma, J_i \quad (\text{all } i\in I)}{\Gamma, \wedge_{i\in I}J_i}$$ ***(conj.: for all i∈I, and rec. in i)***

Remark the asymmetry with ∨: we do not have $\Gamma, \wedge_{i\in I}J, \Delta$

Unfolding the rules of PA_ω for atomic formulas

If we replace $\vee_{i\in I}A_i$, $\wedge_{i\in I}A_i$ in the rules of **PA_ω** with a false, true atomic formula (hence I=∅) we obtain no rule in the disjunctive case, and in the conjunctive case a rule with no assumptions:

$$\frac{}{\Gamma, a}$$ ***(a atomic true)***

Unfolding the rules of PA_ω for $\wedge$, $\vee$

If we replace $\vee_{i\in I}A_i$, $\wedge_{i\in I}A_i$ in the rules of PA_ω with a finite conjunctive, disjunctive formula (hence I={1,2}) we obtain

$$\frac{\Gamma, A\vee B, \Delta, A}{\Gamma, A\vee B, \Delta}\ (\textit{disj. 1}) \qquad \frac{\Gamma, A\vee B, \Delta, B}{\Gamma, A\vee B, \Delta}\ (\textit{disj. 2})$$

$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A\wedge B}\ (\textit{conjunctive})$$

Soundness, Completeness and Curry-Howard Isomorphism for PA_ω

Theorem. Let A be any closed arithmetical formula.

1. ***(Soundness and Completeness)*** A formula A is a theorem of PA_ω if and only if Eloise has a recursive winning strategy with backtracking on A.
2. ***(Curry-Howard)*** The recursive winning strategies for Eloise on A are in one-to-one correspondence with recursive cut-free proof-trees of A in HA_ω, and two corresponding trees are tree-isomorphic.

Classically, A is a theorem of if and only if A is classically true. Thus, games with backtracking are a recursive, sound and complete interpretation of classical truth. 51

Unfolding the rules of PA_ω for $\exists$,$\forall$

If we replace $\vee_{i\in I}A_i$, $\wedge_{i\in I}A_i$ in the rules of PA_ω with an existential, universal quantifier (hence I={closed terms}) we obtain:

$$\frac{\Gamma, \exists x.A, \Delta, A[t/x]}{\Gamma, \exists x.A, \Delta}\ (\textit{disjunctive})$$

$$\frac{\ldots\ \Gamma \vdash A[t/x]\ \ldots}{\Gamma \vdash \forall x.A}\ (\textit{conj.: all closed terms t, rec. on t})$$

Talk 3. Stratifying Game Models for Arithmetic through the level of backtracking

The level of backtracking connects winning strategies to subclassical logics and to the degrees of a non-recursive map

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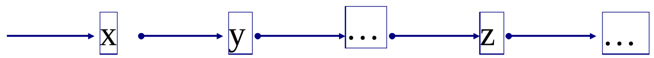
The level 1 of Backtracking

- In the previous lessons we introduced ***"Backtracking"***, the possibility, in a game, to come back to a previous move and retract it, forgetting everything took place after it.
- Backtracking is complex to describe and to implement.
- Coquand ([Coq91], p.90) proposed a simpler notion of backtracking, now called ***"1-backtracking"***. It is a particular case of backtracking in which forgetting is ***irreversible***. If we forget a move we can never restore it back.
- We will characterize the formulas validated by 1-backtracking.

What we know about 1-Backtracking

- **Learning**. 1-backtracking characterizes the set of formulas we can "learn" (in the sense of **LCM, Limit Computable Mathematics**) by Incremental Learning.
- **Recursion Theory.** 1-backtracking ***simulates***, in any computation, an oracle for the Halting Problem.
- **Program extraction.** 1-backtracking interprets as algorithms exactly all classical proofs using Excluded Middle only on Σ^0_1-formulas.
- **Stratification of Backtracking.** To each strategy with backtracking we may assign a level of backtracking, ranging from 1 to any recursive ordinal α.

§ 0. The most general notion of Games in Set Theory



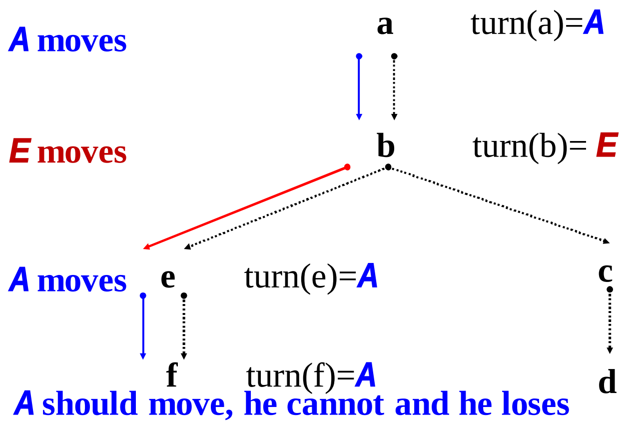
Games in Set Theory

A game **G = (T, R, turn, W_E , W_A)** between two players, **E** (Eloise or Myself) and **A** (Abelard or Nature), consists of:

1. a tree **T** of positions of the game, with a father/child relation **R**.
2. A map **turn**:T→{**E**, **A**}.
3. A partition **(W_E, W_A)** of infinite branches of T.

A play is any finite or infinite branch of T, starting from the root of T. In each node x of the branch, the player **turn**(x) must select a child of x in T, ***otherwise his opponent wins***. If a play continues forever, the winner is **E** if the play is in **W_E**, and the winner is **A** if the play is in **W_A**.

An example of Game and of play



Tarski Games are a particular case of Set-Theoretical games

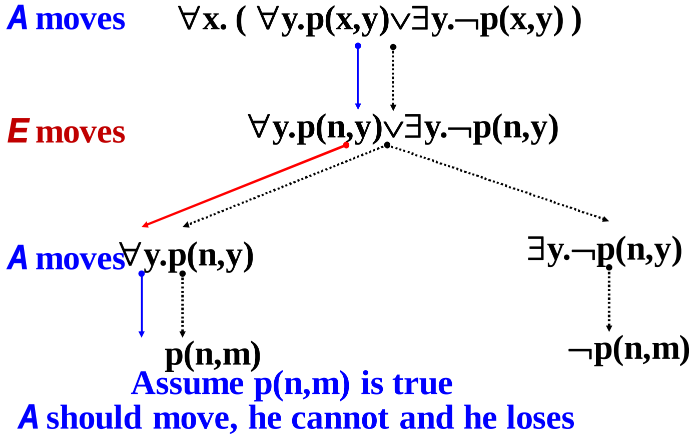
- Fix any arithmetical formula **A**.
- **E** argue in favor of the truth of **A**, and **A** argue in favor of the falsity of **A**.
- If **A** is disjunctive, then **E** must pick some immediate subformula **A'** of **A**, and argue in favor of the truth **A'**. If **A** is conjunctive, then **A** must pick some immediate subformula **A'** of **A**, and argue in favor of the falsity of **A'**.
- When **A** is atomic or atomic negated, if **A** is true then **E** wins, if **A** is false then **A** wins.

Formal definition of Tarski Games

1. Tree **T** of positions = subformula tree of **A**.
2. **turn**(B)=**E** if $B=\exists xC, C\vee D$ or $B=p(t_1,\dots,t_n)$, $\neg p(t_1,\dots,t_n)$ for some p recursive, and B is false. **turn**(B)=**A** if $B=\forall xC, C\wedge D$ or $B=p(t_1,\dots,t_n)$, $\neg p(t_1,\dots,t_n)$ for some p recursive, and B is true.
3. **W_E = W_A = ∅** (T has no infinite branch)

Let $B = p(t_1,\dots,t_n)$, $\neg p(t_1,\dots,t_n)$. If B is false, then **E** should move, she cannot and she loses. If B is true, then **A** should move, he cannot and he loses.

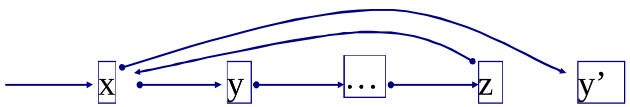
The Tarski's Game for EM-1
Excluded Middle for Σ^0_1-formulas



Winning strategies

- A ***winning strategy*** (for **E**) is any map taking any position of the game from which **E** moves, and returning some move for **E**. We already proved:
- **Thm. E** has a winning strategy in the Tarski game **G** for **A** if and only if **A** is true.
- **We already remarked that u**sually (for instance, when A=1-EM) a winning strategy for **G** is not recursive.

§ 1. 1-Backtracking as an operator on Set-Theoretical Games



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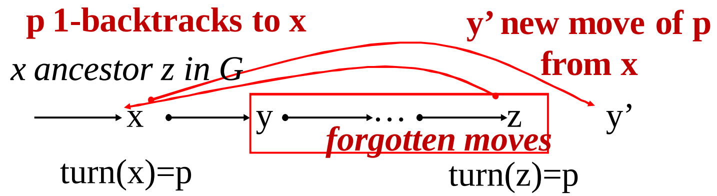
Games with 1-backtracking

- Given a game G, we define a game **bck(G)**. In bck(G), any player **p** moving from a position z can come back to some previous position x, provided: ***x is an ancestor of z in the tree of positions of G***, and ***p moved from x***. Then p definitively erases all moves after x, and he makes a new move y'.



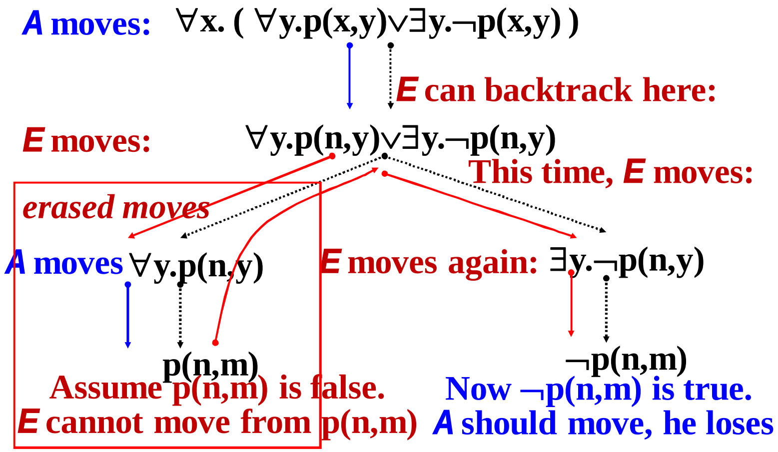
1-backtracking is irreversible

- We said that if a player p comes back to a position x, all moves after x are ***``irreversibly erased''*** from the memory of the other player.
- The player who erased them may still see them, but cannot come back to them, because this would make them again visible to his opponent.
- In general (unlimited) backtracking there is no irreversible erasing. If a player may see some moves, he may come back to them, showing them again to his opponent.

Infinite 1-backtracking is losing

- It is still valid the rule that a player p is allowed to backtrack to a given position x_i of the play ***only finitely many times***. Intuitively, backtracking is a way, for p, of ***``learning the best move''*** from x_i, but we only allow a finite time to learn.
- A player p backtracking infinitely many times to the same position x_i, in violation of the constraint above, loses.
- If E backtracks infinitely many times to some x_i and A infinitely many times to some x_j, the loser is the player backtracking infinitely many times to a position with smaller index.

An example of 1-backtracking for E



The intuition behind 1-backtracking

- If a player is allowed to 1-backtrack, we imagine he may **find out for sure** that a move is wrong. Whenever he **finds a wrong move**, he can come back to it, irreversibly erase it from the memory of its opponent, and make a different move.
- We only allow ***a finite time to learn***. After finitely many mistakes, a player should select some definitive move, otherwise he loses.
- We will formally define a game **bck(G)**, associated to G, in which ***both player can learn ``better'' moves in the game G***.

An operator removing 1-backtracking

- Fix any (finite or infinite) play **σ** = <s_0, …, s_n, …> of bck(G). We can remove 1-backtracking from σ, ***by waiting that both players stop erasing moves***. The result is some canonical (finite or infinite) backtracking-free play **$\sigma^{(1)}$** = <t_0, …, t_n, …> in the ***original game G***.
- Definition of $\sigma^{(1)}$ runs as follows:

1. t_0 = initial position of G.
2. t_{n+1} = last child of t_n in σ, provided t_n has a last child in σ. Otherwise $\sigma^{(1)}$ ends.

$\sigma^{(1)}$ is not recursive in general

Formal definition of the game bck(G)

- **Nodes of bck(G).** All finite successions <s_0, …, s_n> of positions of G, such that: ***(i)*** s_0 initial position of G ***(ii)*** any s_{i+1} is a child in G of some s_j, with j≤i, ***s_j ancestor in G of s_i***, and **turn**(s_j) = **turn**(s_i).
- **Turn.** The player moving from a node <s_0, …, s_n> of bck(G) is the player moving from s_n.
- **Winner of a infinite play.** The winner of an infinite play σ in bck(G) is the winner, in G, of the backtracking-free play $\sigma^{(1)}$.

§ 2. What we know about 1-Backtracking

∃, ∨, ∀, ∧ | *Tarski(A)* | *bck(Tarski(A))*

HA + ω-rule + 1-EM

§2 Limit Computable Mathematics

- The notion of learning in the limit or incremental learning is due to Gold.
- S. Hayashi formalized it by a Realization model having, as realizers, all maps ***recursive in an oracle for the Halting problem***. These maps are, equivalently, all Δ^0_2 maps.
- Total realizers interpret proofs, and are exactly all total ***recursive limits of recursive maps***.
- The set of arithmetical formulas realizable in this model is called Limit Computable Mathematics or LCM.

§2.2 Limit Realization

Denote with {a}'(x) the result of the application of the a^{th} partial map in Δ^0_2 to x. We define a|=A, or "a realizes A" by induction on A:

- **a |= (s = t)** ⇔ (s = t)
- **a |= A∧B** ⇔ (p_1(a) |= A) ∧ (p_2(a) |= B)
- **a |= A∨B** ⇔ (p_1(a) = 1 ∧ p_2(a) |= A) ∨ (p_1(a) = 2 ∧ p_2(a) |= B)
- **a |= A⇒B** ⇔ ∀x∈N. (x |= A) ⇒ ({a}'(x) |= B)
- **a |= ∀x.A** ⇔ ∀x∈N. ({a}'(x) |= A)
- **a |= ∃x.A** ⇔ p_2(a) |= A[x/p_1(a)]

§2.3 Realization and Limit Realization

- The difference with the standard Realization interpretation is that we consider an enumeration {.\}'(.) of all partial maps recursive in the Halting problem, instead of all partial recursive maps.
- **LCM** is not defined giving an axiomatization, but through a semantics, as the set of realizable formulas.
- Most theorems of a first course in Algebra are learnable in the limit, while some crucial theorems of a first course in Analysis (like the compactness of an interval [a,b]) are not (see LICS04, APAL06).

1-Backtracking and Limit Computable Mathematic

- Let A be any arithmetical formula in the connectives ∃, ∨, ∀, ∧. Let Tarski(A) be the Tarski game associated to A. Let LCM be Hayashi's Limit Computable Mathematic (or ***"Arithmetic with incremental learning"***).
- **Theorem.** A is realizable in LCM if and only E if has a recursive winning strategy on bck(Tarski(A)).

1-backtracking characterizes the set of formulas we can "learn" incrementally

1-Backtracking and Recursive Degrees

- Let G any game either ***with alternating players***, or ***with no infinite plays***. Let p any player.
- **Theorem.** p has a winning strategy of recursive degree 1 on G if and only if p has a winning strategy of recursive degree 0 on bck(G).

1-backtracking can replace, in a game strategy, an oracle for the Halting Problem

1- Backtracking and Excluded Middle

- Let A be any arithmetical formula in the connectives ∃, ∨, ∀, ∧. Let Tarski(A) be the Tarski game associated to A. Let HA be Intuitionistic Arithmetic. Let 1-EM be Excluded Middle for degree 1 formulas.
- **Theorem.** E has a recursive winning strategy for bck(Tarski(A)) if, and only if:

HA + ω-rule + 1-EM |- A

1-backtracking interprets as algorithms exactly all classical proofs using only 1-EM

Cut-free 1-Backtracking

- As done by Coquand for general backtracking, we can define a *cut-free* version $bck_{CF}(G)$ of bck(G).
- $bck_{CF}(G)$ is the subgame of bck(G) in which **A** ***cannot backtrack*** (**A** cannot answer to a move which is not the previous one: he cannot ``learn”).
- It is much easier to define winning strategies for **E** on $bck_{CF}(G)$ than on bck(G), because **A** has a serious handicap in $bck_{CF}(G)$.
- As in the case of general backtracking, every winning strategy for **E** on $bck_{CF}(G)$ ***can be raised***, in a canonical way, to a winning strategy for Eloise on bck(G) (in which **A** has no handicap, and can ``learn’’).

2-Backtracking

- We defined the maps G|→bck(G),$bck_{CF}(G)$ from games of Set Theory to games of Set Theory.
- ***By iteration***, for any n∈N we can define $bck^n(G)$, $bck_{CF}{}^n(G)$, the games with *n-backtracking* over G, with and without cuts.
- The difference between 1-backtracking and 2-backtracking is that, in 2-backtracking, ***forgetting is sometimes reversible*** (by 2-backtracking, we may recover a previous move forgotten by 1-backtracking).

Higher levels of Backtracking

- By direct limit we can define $bck^\alpha(G)$, $bck_{CF}{}^\alpha(G)$, for all ordinal α.
- As the superscrpt α increases, ***forgetting becomes more and more reversible*** in the game $bck^\alpha(G)$.
- 2-Backtracking validates exactly the theorem of Heyting Arithmetic with rec. ω-rule and EM-2: this part of the classical logic is required for elementar analysis
- 3-Backtracking validates exactly the theorem of Heyting Arithmetic with rec. ω-rule and EM-3: this part of the classical logic is required for Ramsey Theorem and most of its corollaries.

A stratification of unlimited Backtracking

- For any game G *with alternating players and all plays of length ≤ n*, Coquand defined a game Coq(G). In Coq(G), Eloise has an ***unlimited*** backtracking over G. Abelard, instead, cannot backtrack: Coq(G) is cut-free.
- **Theorem.** $bck_{CF}^{1}(G)$ is a subgame of Coq(G), and conversely, any winning strategy in Coq(G) is a winning strategy in some $bck_{CF}^{\alpha}(G)$.

Unlimited backtracking can be obtained by iterating 1-backtracking.

Talk 4. A definition of truth for Intuitionistic Arithmetic through Games with Backtracking

An intutionistic proof uses no backtracking at all in output and unlimited backtracking on inputs

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Abstract of the Talk

1. There is common agreement that the first order intuitionistically valid formulas are those derivable in first order intuitionitistic logic LJ and validated by a suitable **Kripke model [ref].**
2. There is no common agreement to what the notion of truth for intuitionistic first order arithmetic should be. Is the **Markov principle** intuitionistically true? Is the **negation of Excluded Middle** intuitionistically true?
3. In this talk we introduce a **game theoretical model with backtracking** of first order Arithmetic which, we claim, validates exactly the intuitionistically true formulas.

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A notion of Truth for Intuitionism

1. We consider the intuitionistically true formulas as the classically true formulas in which **if we prove $\exists x P(x,y)$** we have a way of computing in finite time some **x and some proof of P(x,y)** out of y, and if we prove **$A_1(y) \vee A_2(y)$** we have a way of computing in finite time some **i=1,2 and a proof of A_i** out of y.
2. This rules out the BHK Realization semantics [ref], which validate the negation of Excluded Middle and therefore are not included in Classical Logic.
3. The values **x, i are computable in game semantics** with backtracking [Coq95], but **sometimes they are wrong** and must be changed.

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Which notion of Truth for Intuitionism?

1. We will restrict backtracking semantics in such a way that **once we send a value x,i then we cannot change it**: this is our interpretation of ***"x,i are computable".***
2. In order to represent proofs of implications A→B as functions, we represent negative formulas A as input gates and positive formulas (the only formulas considered in [Coq95]) as output gates.
3. This game model is influenced from Hyland-Ong game model of lambda calculus [ref] in the representation of negative/positive formulas, in the representation of disjunctive/conjunctive formulas by Coquand.

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Intuitionistic Truth and backtracking

1. We consider **game with turn conflicts**, and a more general backtracking: **Sequential Backtracking.**
2. We start from **Tarski games**, and we define a game semantics for classical arithmetic with **implication as a primitive connective.** We obtain a game semantics for intuitionism by forbidding to send an output twice.
3. We will prove that there is a one-to-one correspondence (a kind of **``Curry Howard'' isomorphism**) between: proofs of Intuitionistic Arithmetic HA extended with recursive ω-rule, and the winning strategies for games with sequential backtracking for intuitionistic arithmetic.

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The state of the art.

- In 1975, Lorentzen defined the first sound and complete game semantics for Infinitary Intuitionistic Arithmetic **HA_w** (for details we refer to [Fel]).
- Lorentzen interprets cut-free proofs with infinitary ω-rule of an arithmetical formula A as recursive winning strategies for a game associated to A.
- His game semantics is ***"ad hoc"***, carved on the notion of constructive proofs he wants to interpret. Thus, it is difficult to argue that Lorentzen model is a definition of arithmetical truth. We will show that a game semantics equivalent to **Lorentzen** may be obtained from a game semantics with backtracking for classical logic, by forbidding to change an output after we sent it.

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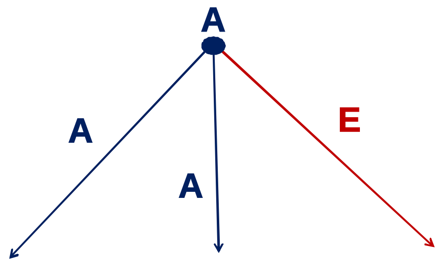
What are Games with turn conflicts

- There are two players, **E (Eloise)** and **A (Abelard)**.
- The set of rules for a game G with turn conflicts is a tree with **nodes and edges having the color** either of **E** or of **A**. Nodes are positions of the game, edges are moves.
- The play starts at the root of G. At each turn, a player may: either **drop out** and lose the game, or move from the current node **along an edge of his color**, or **wait** for his opponent's move.
- If **both E** or **A** want to move, or **both** want to wait, we say there is a **turn conflict**. In this case, the player having the color the node **succumbs**, and **change its choice**.

An example of turn conflict



Both **E** or **A** may move from a node having the color of **A**. If both want to move, **A** waits and **E** moves. If both want to wait, **A** moves and **E** waits. **A** is the player having the color the node, the **succumbing player**, therefore he is forced to **change its choice**.

Winner of a game

- In any **leaf** of G there are no moves left for both players: the succumbing player is **forced to drop out**.
- The player who drops out loses.
- If G is a **finite game** (all branches of G are finite), we decide in this way the **winner for all plays**.
- Otherwise there are infinite plays. In this case, **G** is equipped with two **disjoint** sets of infinite plays: $\mathbf{W_E}$ and $\mathbf{W_A}$.
- **E** wins if the infinite play is in $\mathbf{W_E}$, and **A** wins if the infinite play is in $\mathbf{W_A}$. Otherwise both lose.

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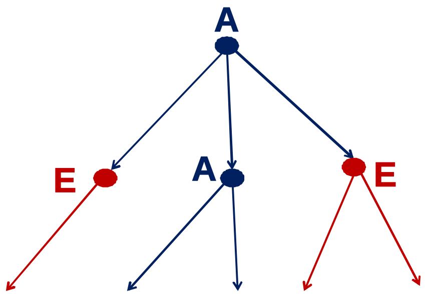
Games without conflict

- If all edges from a node **have the color of the node**, then it is never possible for both players to move from the same node: hence there are never conflicts.
- In this case the **succumbing** player, having color of the node, is the player **forced to move or to drop out** (having the move obligation).
- If there are no turn conflicts, the color of the edge is useless.
- When all edges have the same color of the initial node of their edge, we obtain the usual notion of game, **without turn conflicts.**

Games without turn conflict



When all edges have the same color of the initial node of the edge, we obtain the usual notion of game, **without turn conflicts.**

Backtracking simplifies strategies

- Recall that winning strategy for a game G are often non-recursive, even when G is a recursive tree. If we allow **E** to **retract finitely many times** her move, many winning strategies for **E** become **recursive**. In fact, winning strategies for **E** become programming learning the correct move **by trial and error**.
- We may extend any game **G** with conflict with the possibility for **E** of retracting any previous move.
- This notion of backtracking is **broader that in [Coq95]**: we call it **G with Sequential Backtracking** or **Seq(G).** Seq(G) **always has turn conflicts**, even if G had no turn conflicts.

A new notion of game: Seq(G)

- The **color of a node** in Seq(G) is the same as in G.
- The moves of **A** in Seq(G) and in G are the same.
- **E** may move **from any position** in Seq(G) (even if his opponent should move), and has two kinds of possible moves.

1. ***Explicit Backtracking (new)* . E** may come back to any previous node in the history of the play, then **E duplicates** it as next move
2. ***Implicit Backtracking (as in [Coq95]).* E** may come back to any previous node in the history of the play from which **E** may move, then **E** produces **a move in the original G** from it as next move.

The winner of an infinite play

- We include here the winning condition for infinite plays of Seq(G) only in the case G is **a finite play and there is no cut rule**. In this case we ask: all infinite plays in Seq(G) are **won by A**.
- Why? In Seq(G), **E** is allowed to retract finitely many times her previous move, but only **in order to find a better move** by trial-and-error.
- If G is a finite play, a play in Seq(G) is infinite only if **E changes infinitely many times her move** from a given node, just to waste time and to avoid losing the game.
- This behavior is unfair and therefore is **penalized**: **E** loses any infinite play.

§2. Tarski games with Sequential Backtracking

- Tarski games are the canonical notion of games (without turn conflicts) representing the truth of an arithmetical statement. In order to define Tarski games, we consider a first order language **L: True, False, $\vee$, $\wedge$, $\neg$, $\rightarrow$,** $\forall$, $\exists$, with **all connectives** and **all primitive recursive** predicates and functions.
- We define a relation $<_1$ (immediate subformula) for closed formulas of **L**. We set $A <_1 \neg A$ and:

$$A, B \;<_1\; A\vee B,\; A\wedge B,\; A\rightarrow B$$

$$A[t/x] \;<_1\; \forall x.A, \exists x.A$$ ***(for all closed terms t)***

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Game theoretical meaning of Disjunctive, conjunctive, positive and negative formulas

- $A\vee B$, $\exists x.A$, $A\rightarrow B$, $\neg A$ are **disjunctive** formulas.
- $A\wedge B$, $\forall x.A$ are **conjunctive** formulas.
- $A <_1 A\rightarrow B$, $\neg A$ is a **negative** subformula. In all other cases $A <_1 C$ is a **positive** subformula.
- ***Negative*** formulas are **input gates.** ***Positive*** formulas are **output gates**.
- ***Disjunctive positive*** formulas correspond to **sending an output**, ***conjunctive positive*** formula to **receiving an output**. ***Disjunctive negative*** formulas correspond to **sending a input**, ***conjunctive negative*** formula to **receiving a input**.

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Disjunctive, conjunctive, positive and negative "judgements"

- **Judgements:** J = s.A, where either s=true or s=false.
- **true.A** is a positive judgement. **true.A** is disjunctive (conjunctive) iff A disjunctive (conjunctive).
- **false.A** is a negative judgement. **false.A** is disjunctive (conjunctive) iff A conjunctive (disjunctive).
- **s.A$<_1$t.B** if and only if: A $<_1$ B, and s=t if A is a positive subformula of B, and s≠t if A is a negative subformula.
- For instance, **false.A, true.B $<_1$ true.A→B**.
- We write a conjunctive judgement **J** as $\wedge_{i\in I}J_i$ for all $J_i <_1 J$, and a disjunctive judgement **J** as $\vee_{i\in I}J_i$ for all $J_i <_1 J$. The result of $\wedge_{i\in I}J_i$, $\vee_{i\in I}J_i$ is unique.

The game Tarski(s.A)

- We write ≤ for the transitive closure of $<_1$. For each judgement s.A we define **Tarski(s.A)**, the game associated to the notion of truth for s.A.
- The nodes of Tarski(s.A) are all judgements t.B ≤ s.A. The **root** is s.A, the **child/father** relation is t.B $<_1$ u.C.
- Disjunctive formulas and edges from them are **colored E**, conjunctive formulas and edges from them are **colored A**. We recall the following:
- **Theorem (Completeness for Tarski games and Truth). E** has an **arithmetical** winning strategy from Tarski(s.A) if and only if s.A is classically true.

The game model Int(s.A) for intuitionistic truth

We write <, ≤ for the strict and large order on judgements associated to $<_1$. For each judgement s.A we define **Int(s.A)** by **a restriction of the backtracking** in Seq(Tarski(s.A)).

(1) Ordinary move. $\mathbf{m_k}$ = **t.B**, for some t.B ≤ s.A.

(2) Backtracking or duplication move. $\mathbf{m_k}$ = **bck(i)+t.B** for some i∈N and some t.B ≤ s.A.

(3) End move. $\mathbf{m_k}$ = **drop**. *"I quit"*.

Positions. The position associated to m = t.B, or to m = (bck(i) + t.B) is t.B.

Players and turn for Int(s.A)

The Players are: *Eloise* (defending the truth of the current position t.B of the play), ***Abelard*** (attacking the truth of the current position t.B of the play).

The player moving from a position t.B of the play is defined as follows.

1. If t.B is a disjunctive judgement, then ***Eloise*** moves next.
2. If t.B is a conjunctive judgement, then ***Abelard*** moves next, ***unless Eloise asks to move next*** and she moves bck(i)+t.B.

We record, for each position, the player moving next as part of the history of the play.

Plays for the cut-free game Int(s.A)

Inductive definition of Plays of G(t.A).

1. p = s.A is a play.
2. Assume $p=m_1, \ldots, m_k$ is a play. Let $s_i.A_i$ be the position associated to m_i, for any $1 \le i \le k$.

- **Correct ordinary move from p:** m_{k+1}=some $t.B <_1 s_k.A_k$
- **Correct backtracking move from p:** m_{k+1} = bck(i)+t.B for some $1 \le i \le k$, some $t.B \le_1 s_i.A_i$, provided:

(1) Eloise moves next from $s_k.A_k$ ***(only Eloise backtracks)***

(2) Either $t.B =_1 s_i.A_i$, or $t.B <_1 s_i.A_i$ and $s_i.A_i$ is disjunctive.

(3) if s_i = true, then $s_i.A_i$ is the **last positive position** of p.

The Winner of a play of Int(s.A)

- A play of G(s.A) is **terminated** if either it is finite and ending with drop, or it is infinite. **If a terminated play ends with "drop"**, the player playing "drop" loses.
- **If a terminated play is infinite**, in a cut-free game the winner is Abelard.

Why are infinite plays lost by Eloise? Eloise, to avoid losing the game, may come back infinitely many times to the same position, just to waste time. This behaviour is unfair and therefore it is penalized.

Discussing the rules of Int(s.A)

For any backtrack/duplication move m_{k+1} = bck(i)+t.B we asked that $1 \le i \le k$, that $t.B \le_1 s_i.A_i$, and three clauses.

(1) "Eloise moves next from $s_k.A_k$" This means that ***only Eloise may backtrack*** (come back to a previous position).

(2) "Either $t.B =_1 s_i.A_i$, or $t.B <_1 s_i.A_i$ and $s_i.A_i$ is disjunctive." Thus, the only way for Eloise to come back to a conjunctive position $s_i.A_i$, is to move bck(i)+$s_i.A_i$, duplicating the conjunctive position $s_i.A_i$. ***Backtracking to disjunctive and conjunctive positions is asymmetric.***

(3) "If s_i = true, then $s_i.A_i$ is the last positive position of p" ***We forbid to change an output after we sent it***.

A formulation of HA_w satisfying the proof/strategy isomorphism

- The language of $\mathbf{HA_\omega}$ are all judgements. Any judgement is of the form $\vee_{i\in I} J_i$ or $\wedge_{i\in I} J_i$. Say: **true.A→B = ∨{false.A,true.B}** and **false.A→B=∧{true.A, false.B}**.
- Sequents of $\mathbf{HA_\omega}$ are **ordered lists** of judgements. Therefore Contraction and Exchange rules are **not built-in** in the notion of sequent.
- We explicitly assume Contraction in $\mathbf{HA_\omega}$. We hyde Exchange rule through the fact that the active formula, if disjunctive, may be in any position in the sequent.
- Identity rule is trivially derivable in $\mathbf{HA_\omega}$. **Cut rule is derivable but not trivial at all.**

A formulation of HA_ω with 3 rules (in one-side form, with judgements)

$\dfrac{\Gamma, \vee_{i\in I}J_i, \Delta, J_i}{\Gamma, \vee_{i\in I}J_i, \Delta}$ *(disj. with implicit contraction and exchange: for some $i\in I$)*

provided that: if $\vee_{i\in I}J$ is positive, then $\vee_{i\in I}J$ is the last positive judgement in $\Gamma, \vee_{i\in I}J, \Delta$.

$\dfrac{\Gamma, \wedge_{i\in I}J_i, J_i \quad (\text{all } i\in I)}{\Gamma, \wedge_{i\in I}J_i}$ *(conj. with implicit contraction: for all $i\in I$, and rec. in i)*

Remark the asymmetry with $\vee$: we do not have $\Gamma,\wedge_{i\in I}J,\Delta$

$\dfrac{\Gamma, J, \Delta, J}{\Gamma, J, \Delta}$ *(contraction with implicit exchange)* **provided that:** if J is positive, then J is the last positive judgement in Γ, J, Δ.

Any one-sided sequent represents a two-sided intuitionistic sequent

- Any one-sided ordered **list** of judgements has the form

$$\Gamma = \text{false}.\Gamma_0, \text{true}.B_1, \text{false}.\Gamma_1, \ldots, \text{true}.B_n, \text{false}.\Gamma_n$$

for some ordered **lists** of negative judgements of the form:

$$\text{false}.\Gamma_i = \text{false}.A_{i,1}, \ldots, \text{false}.A_{i,m_i}$$

- Γ represents (many-to-one) the following two-sided intuitionistic ordered sequent of formulas:

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_n \vdash B_n$$

- ***The formulas $B_1, \ldots, B_{n-1}$ are not visible in the two-sided form of Γ.*** The 3 rules of HA_ω correspond, in the two-sided version, to one rule for each connective and side.

Unfolding the rules of HA_ω for $\rightarrow$

If we replace the one-side sequent of judgements with the corresponding two-side sequent of formulas, then in the case of **A→B** we obtain:

$\dfrac{\Gamma, A \vdash A\rightarrow B}{\Gamma \vdash A\rightarrow B}$ *(disjunctive 1)* $\qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A\rightarrow B}$ *(disjunctive 2)*

$\dfrac{\Gamma, A\rightarrow B \vdash A \qquad \Gamma, A\rightarrow B, B \vdash D}{\Gamma, A\rightarrow B \vdash D}$ *(conjunctive)*

$\dfrac{\Gamma, A\rightarrow B, \Delta, A\rightarrow B \vdash D}{\Gamma, A\rightarrow B, \Delta \vdash D}$ *(contraction+exchange)*

Unfolding the rules of HA_ω for $\forall$

If we replace the one-side sequent of judgements with the corresponding two-side ordered sequent of formulas, then in the case of **∀x.A** we obtain:

$\dfrac{\Gamma, \forall x.A, \Delta, A[t/x] \vdash D}{\Gamma, \forall x.A, \Delta \vdash D}$ *(disjunctive)*

$\dfrac{\ldots\ \Gamma \vdash A[t/x]\ \ldots}{\Gamma \vdash \forall x.A}$ *(conj.: all closed terms t, rec. on t)*

$\dfrac{\Gamma, \forall x.A, \Delta, \forall x.A \vdash D}{\Gamma, \forall x.A, \Delta \vdash D}$ *(contraction+exchange)*

Isomorphism and Cut-Elimination Theorem for HA$_\omega$

Theorem. Let A be any closed arithmetical formula.

1. ***(Soundness and Completeness)*** A formula A is a theorem of **HA$_\omega$** if and only if Eloise has a recursive winning strategies on the cut-free game Int(true.A).
2. ***(Curry-Howard)*** The recursive winning strategy-trees for Eloise on Int(true.A) are tree-isomorphic to the infinitary recursive cut-free proof-trees of A in **HA$_\omega$**.
3. ***(Cut-Elimination)*** It is translated in a game-theoretical result: "any dialogue of two terminating strategies is terminating". *Asymmetry of backtracking to disjunctive and conjunctive positions is essential for termination.*

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