note
A Kripke Model for Simplicial Sets

Marc Bezem* Thierry Coquand†

Abstract

By means of a countermodel we show that the homotopy equivalence of the fibers of a Kan fibration over a connected base cannot be proved constructively.

1 Introduction

It is generally assumed that Kan fibrations in the category of simplicial sets are inherently 'non-constructive' since important properties of Kan fibrations have only been proved using classical logic. This has important consequences in that Voevodsky's model construction for type theory with the Univalence Axiom [1] cannot be internalized in type theory, thus blocking one solution of the fundamental problem of the computational interpretation of univalence. Another consequence has been that model constructions based on semi-simplicial sets have been considered, for example, in [2], but these come with a separate set of challenges. However, not knowing how to prove something constructively leaves open the possible existence of a constructive proof. In this note we show that a constructive proof of one of the basic properties of Kan fibrations cannot exist. We consider this formal unprovability result as a necessary first step towards a constructive reformulation of Kan simplicial set theory.

2 Preliminaries

We assume familiarity with the notions (*opposite*) *category*, *functor* and *natural transformation*, which can be found in, for example, [4]. The category Δ consists of objects $[n] = \{0, \ldots, n\}$ for every $n \in \mathbb{N}$, equipped with the standard ordering, and order-preserving maps. A *simplicial set* is a functor from Δ^{op} to **Set**.

The simplicial set Δ^k, the standard k-simplex, is defined by $\Delta^k[n] = [n] \to [k]$ with $u \longmapsto u \circ f$, $\Delta^k[n] \to \Delta^k[m]$, for all order-preserving $f : [m] \to [n]$. The simplicial set Λ^k_j, the j-th horn of the standard k-simplex, is defined by $\Lambda^k_j[n] = \{f \in \Delta^k[n] \mid [k] - \{j\} \not\subseteq Im(f)\}$. The simplices of Λ^k_j are obviously closed under precomposition with any order-preserving map $[m] \to [n]$. See [5, 6] for more information, in particular on the simplicial identities and the decomposition of order-preserving maps in *face maps* $d_i : [n] \to [n+1]$ (order-preserving injections skipping i) and *degeneracy maps* $s_j : [n+1] \to [n]$ (order-preserving surjections repeating j).

Simplicial sets with natural transformations as maps form a category. The following maps of simplicial sets are used in the sequel, and may serve as (trivial) examples: the embedding $e : \Lambda^k_j \to \Delta^k$; the constant map $c_i : X \to \Delta^k$ with $c_i[n](x) : [n] \to [k] : m \longmapsto i$, for all $x \in X[n]$ $(0 \leq i \leq k)$.

A *Kan fibration* is a map $p : X \to Y$ of simplicial sets that satisfies a particular extension condition. We briefly summarize the definition and refer for more explanation to [5, Definition 7.1] or [6, I.3]. The map p is Kan if for every horn Λ^k_j and all maps $f : \Lambda^k_j \to X$ and $g : \Delta^k \to Y$ such that $p \circ f = g \circ e$, there exists a map $h : \Delta^k \to X$ such that $f = h \circ e$ and $g = p \circ h$.

The phenomenon we study, the undecidability of degeneracy, has already important consequences in dimension 1. This makes it possible to restrict attention to dimensions 0 and 1, where simplicial sets have a multigraph structure. Doing so simplifies the presentation and actually gives a stronger counterexample.

*Department of Informatics, University of Bergen, `bezem@ii.uib.no`
†Department of Computer Science and Engineering, Chalmers/University of Gothenburg, `coquand@chalmers.se`

Also, it suffices to work with Kan fibrations with two fibers. The following two definitions specify the graph structures, and Definition 2.3 the corresponding simplicial sets and Kan fibrations.

Definition 2.1 A *reflexive multigraph* consists of C_1, C_0, d_0, d_1, s where C_0 is a set of points, C_1 a set of edges, $d_i : C_1 \to C_0$, d_1 the *source* and d_0 the *target* function, and $s : C_0 \to C_1$ the function mapping each c to a selfloop of c. We write $e : a \to b$ if e is in C_1 such that $d_1(e) = a$ and $d_0(e) = b$ (note the direction!). In particular we have $d_i(s(c)) = c$ for all $c \in C_0$. A *Kan graph* is a reflexive multigraph having the following explicit filling operation fill_2: for all a, b, c in C_0 and $e : a \to b$ and $f : a \to c$ we have such that $\mathit{fill}_2(e, f) : b \to c$ in C_1.

As a consequence of the above definition, $\mathit{fill}_2(e, s(a)) : b \to a$ for all $e : a \to b$, so a Kan graph is symmetric. If we also have $f : b \to c$, then $\mathit{fill}_2(\mathit{fill}_2(e, s(a)), f) : a \to c$, so a Kan graph is transitive. Kan graphs are precisely the reflexive, symmetric and transitive multigraphs with explicit operations.

The use of explicit operations, here and below, avoids unnecessary appeals to the Axiom of Choice. More importantly, it pre-empts the objection that the counterexample can be given because the input data lacks essential information.

Definition 2.2 A *Kan Δ^1-graph* is defined by the following data (intuitive explanation below):

1. Two Kan graphs A_0, A_1 and B_0, B_1 with their respective maps d_i, s and fill_2.
2. A set G and two maps $d_1 : G \to A_0$ and $d_0 : G \to B_0$. Again we write $e : a \to b$ if e is in G such that $d_1(e) = a$ and $d_0(e) = b$ (no confusion will arise from using the same notation).
3. The following filling operations: for all $a \in A_0$ we have $\mathit{fill}_1(a) : a \to b$ in G, with $b = d_0(\mathit{fill}_1(a)) \in B_0$; for all $b \in B_0$ we have $\mathit{fill}_1(b) : a \to b$ in G, with $a = d_1(\mathit{fill}_1(b)) \in A_0$.
4. The following filling operations: for all $a \in A_0$, $b \in B_0$, $c \in A_0 + B_0$ and $e : a \to b$ in G and $f : a \to c$ in $A_1 + G$, there exists $\mathit{fill}_2(a, f, e) : c \to b$ in $G + B_1$; for all $a \in A_0$, $b \in B_0$, $c \in A_0 + B_0$ and $e : a \to b$ in G and $f : c \to b$ in $G + B_1$, there exists $\mathit{fill}_2(e, f, b) : a \to c$ in $A_1 + G$.

In fact, the Kan graph property of A and B can be derived from the last three clauses above. The intuition behind the definition of Kan Δ^1-graph is: A represents the fiber over 0, B the fiber over 1, and G represents the liftings of $(0 \to 1) = id_{[1]} \in \Delta^1[1]$ to the fibers A and B. (The direction of the edges in G is consistent with $0 \to 1$.) Note that the subscripts in A_0, A_1 and B_0, B_1 refer to the dimension.

In the following definition we construct the canonical simplicial set and the canonical Kan fibration implicit in the data of Definition 2.2, validating the intuition.

Definition 2.3 Let data be as in Definition 2.2. Define the simplicial set E by $E[0] = A_0 + B_0$, $E[1] = A_1 + G + B_1$ and $E[n]$, for $n \geq 2$, consisting of all objects of the form $(u_0, ..., u_n; \ldots, e_{ij}, \ldots)$ such that there exists a l with $0 \leq l \leq n + 1$ and

$$\begin{aligned}
&e_{ij} : u_i \to u_j \text{ in } A \text{ for all } 0 \leq i < j \leq l - 1,\\
&e_{ij} : u_i \to u_j \text{ in } B \text{ for all } l \leq i < j \leq n,\\
&e_{ij} : u_i \to u_j \text{ in } G \text{ for all } 0 \leq i \leq l - 1,\ l \leq j \leq n \text{ (so } u_i \in A_0,\ u_j \in B_0).
\end{aligned}$$

The maps d_k in E are defined by removing from $(u_0, ..., u_n; \ldots, e_{ij}, \ldots)$ the point u_k and all edges e_{ik} and e_{kj}. The maps s_k in E are defined by duplicating the point u_k in $(u_0, ..., u_n; \ldots, e_{ij}, \ldots)$, adding an edge $e_{k(k+1)} = s(u_k)$, and duplicating edges and incrementing indices of edges as appropriate. This completes the construction of the simplicial set E. The fibration $p : E \to \Delta^1$ such that A and B represent its fibers is simply $p(u_0, ..., u_n; \ldots, e_{ij}, \ldots) : [n] \to [1] : i \longmapsto (0 \text{ if } u_i \in A,\ 1 \text{ if } u_i \in B)$. The fact that p is Kan can be seen as follows. Let Λ^n_k $(n \geq 1)$ be a horn and $f : \Lambda^n_k \to E$ $(0 \leq k \leq n)$. Let $e : \Lambda^n_k \to \Delta^n$ be the embedding and $g : \Delta^n \to \Delta^1$ such that $p \circ f = g \circ e$. We have to define a lifting $h : \Delta^n \to E$. If $n = 1$ we use fill_1 of clause 3 in Definition 2.2. If $n = 2$ we observe that the horn contains all points and we use fill_2 of clause 4 in Definition 2.2. If $n \geq 3$ we observe that the horn contains all points and all edges and we define the lifting by $q \longmapsto (f_{[0]}(q(0)), \ldots, f_{[0]}(q(m)); \ldots, f_{[1]}(e_{ij}), \ldots)$. Here $q : [m] \to [n]$ is order-preserving and e_{ij} is the edge from $q(i)$ to $q(j)$ in $\Delta^n[1] = \Lambda^n_k[1]$, in so far required by E.

Remark 2.4 There is actually a simpler way to extend the data in Definition 2.2 to a simplicial set, namely by adding only degeneracies in higher dimensions. More precisely, define the simplicial set E' by $E'[0] = A_0 + B_0$, $E'[1] = A_1 + G + B_1$ and $E'[n]$, for $n \geq 2$, consisting of all objects of the form $(u_0, ..., u_n; \ldots, e_{i(i+1)}, \ldots)$ such that there exists a l with $0 \leq l \leq n+1$ and $u_i \in A$ for all $0 \leq i < l$, $u_i \in B$ for all $l \leq i \leq n$, and *all but at most one* $e_{i(i+1)}$ degenerated $(0 \leq i < n)$. The corresponding fibration p' is then in general not Kan, and therefore we cannot use this.

Now that we have explained the relation between the Δ^1-graph A, G, B and its Kan fibration $p : E \to \Delta^1$, we can formulate the homotopy equivalence of the fibers of p in terms of A, G, B. Recall that a homotopy equivalence between simplicial sets X and Y consists of maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. For maps $h_0, h_1 : Z \to Z$, homotopy means that there exists a $h : Z \times \Delta^1 \to Z$ such that $h_i = h \circ (id_Z, c_i)$, with c_i the constant map $Z \to \Delta^1$ $(i = 0, 1)$. (Homotopy of simplicial maps need not be symmetric.) Again we refer to [5, Definition 5.1] or [6, I.6] for more explanation. Here only dimensions 0 and 1 matter. For example, $g \circ f$ homotopic to id_A implies that $h[0](a, 0) = g(f(a))$ and $h[0](a, 1) = a$ for all $a \in A_0$. Analyzing $h[1](s(a), 0{\to}1)$ one finds that there must be an edge from $g(f(a))$ to a and vice versa, since Kan graphs are symmetric. This motivates the following proposition.

Proposition 2.5 *Let data be as in Definition 2.2. Then there exist* $f_0 : A_0 \to B_0,\ g_0 : B_0 \to A_0$ *and* $f_1 : A_1 \to B_1,\ g_1 : B_1 \to A_1$ *such that:*

1. *for all* a *in* A_0 *there exists* $u : a \to g_0(f_0(a))$ *in* A_1,
2. *for all* b *in* B_0 *there exists* $v : b \to f_0(g_0(b))$ *in* B_1,
3. *for all* u *in* A_1, $f_0(d_i(u)) = d_i(f_1(u))$ $(i = 0, 1)$,
4. *for all* v *in* B_1, $g_0(d_i(v)) = d_i(g_1(v))$ $(i = 0, 1)$,
5. *(crucial!)* $f_1(s(a)) = s(f_0(a))$ *for all* $a \in A_0$ *and* $g_1(s(b)) = s(g_0(b))$ *for all* $b \in B_0$.

Proof. We present three proofs, all based on classical logic.

First proof. The data in Definition 2.2 defines a Kan fibration $p : E \to \Delta^1$ as in Definition 2.3. Now the proposition follows immediately from the fact that the fibers of 0 and 1 are homotopy equivalent [5, Corollary 7.11].

Second proof. Using clause (3) in Definition 2.2 we define $f_0 = d_0 \circ \mathit{fill}_1$ and $g_0 = d_1 \circ \mathit{fill}_1$. Then we have $\mathit{fill}_1(a) : a \to f_0(a)$ and $\mathit{fill}_1(f_0(a)) : g_0(f_0(a)) \to f_0(a)$, so $\mathit{fill}_2(\mathit{fill}_1(a), \mathit{fill}_1(f_0(a)), f_0(a)) : a \to g_0(f_0(a))$. This proves (1), and (2) is proved similarly.

To define f_1, let $u \in A_1$. We distinguish between u degenerate or not. If u is degenerate, i.e., equal to $s(a)$ for some a in A_0, then $u = s(d_i(u))$ and we define $f_1(u) = s(f_0(d_0(u)))$. Otherwise, using clauses (3) and (4) in Definition 2.2, $e = \mathit{fill}_2(d_1(u), u, \mathit{fill}_1(d_1(u))) : d_0(u) \to f_0(d_1(u))$ and $f = \mathit{fill}_1(d_0(u)) : d_0(u) \to f_0(d_0(u))$, and so $\mathit{fill}_2(d_0(u), e, f) : f_0(d_1(u)) \to f_0(d_0(u))$. It follows that f_1 defined by $f_1(u) = \mathit{fill}_2(d_0(u), e, f)$ satisfies (3). Similarly we can define g_1 satisfying (4). Both f_1 and g_1 satisfy (5) per construction.

Third proof. This proof differs from the previous in that we replace the degeneracy test by the test $d_0(u) = d_1(u)$. In other words, we put $f_1(u) = s(f_0(d_0(u)))$ for *all* selfloops u instead of only for the degenerate one. From the constructive point of view we use a different instance of the Law of the Excluded Middle. But even from the classical point of view the proofs are different: the resulting homotopy equivalences may not be the same. □

The next result is that some use of classical logic is essential in this argument, because of the soundness of Kripke semantics for intuitionistic logic [7].

Proposition 2.6 *The previous proposition does not hold in a Kripke model over the poset* $0 \leqslant 1 \leqslant 2$.

Day	A_0	A_1	G	B_1	B_0
0	$\{a, a'\}$	$\{s(a), s(a')\}$	$\{w : a{\to}b, w' : a'{\to}b'\}$	$\{s(b), s(b'), z : b{\to}b, z' : b'{\to}b'\}$	$\{b, b'\}$
1		$+\{u : a{\to}a', u' : a'{\to}a\}$	$+\{x : a{\to}b', x' : a'{\to}b\}$	$+\{v : b{\to}b', v' : b'{\to}b\}$	
2	$\{a{=}a'\}$	$\{u{=}u'{=}s(a){=}s(a')\}$	$\{x{=}x'{=}w{=}w'\}$	$\{z{=}v{=}v'{=}z'\ ,\ s(b){=}s(b')\}$	$\{b{=}b'\}$

Table 1: Three days in the life of A_0, A_1, G, B_1, B_0 (only what *changes*)

Proof. We shall construct the Kripke model X. The intuition is that X evolves over time as $X(0) \to X(1) \to X(2)$. We can interpret the transition from $X(i)$ to $X(j)$ as adding new elements or equating elements, i.e., extending the equality relation. Table 1 shows A_0, A_1, B_0, B_1, G changing over time.

In words, Table 1 shows how edges are added from day 0 to day 1. From day 1 to day 2, A_0 collapses to one point with all edges degenerated; also B_0 collapses to one point, but the edges z, v, z', v' collapse into one *non-degenerated* self-loop; G collapses to one edge. The filling operations are mostly self-evident, with some notable exceptions. One has to take $\mathit{fill}_2(a, w, w) = z$ and $\mathit{fill}_2(a', w', w') = z'$ from day 0, one cannot use $\mathit{fill}_2(a, w, w) = s(b)$ or $\mathit{fill}_2(a', w', w') = s(b')$ instead. The reason is that $\mathit{fill}_2(a, w, x) = v \neq s(b)$ from day 1, and collapsing on day 2 yields $\mathit{fill}_2(a, w, w) = \mathit{fill}_2(a, w, x)$. For similar reasons, $\mathit{fill}_2(s(b), z) = z$ and not $s(b)$.

All preconditions are now satisfied in the Kripke sense, but there is no way to define f_0, f_1, g_0, g_1 satisfying the required properties. Indeed, the function $f_0(0)$ has to be $a, a' \longmapsto b, b'$ or $a, a' \longmapsto b', b$. In the first case we must have to have $f_1(1)(u) = v$, in the second case $f_1(1)(u) = v'$. But then there is a problem in defining $f_1(2)$ which has to send $s(a)$ both to $s(b)$ and to $v = v'$, see the diagram below. In the following section we describe the formal verification of this proof. $\square$

$$\begin{tikzcd}
 & u & s(a) \\
A_1(0) \arrow[r] \arrow[d, "f_1(0)"] & A_1(1) \arrow[r] \arrow[d, "f_1(1)"] & A_1(2) \arrow[d, "f_1(2)"] \\
B_1(0) \arrow[r] & B_1(1) \arrow[r] & B_1(2) \\
 & v \text{ or } v' & v = v'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & s(a) & s(a) \\
A_1(0) \arrow[r] \arrow[d, "f_1(0)"] & A_1(1) \arrow[r] \arrow[d, "f_1(1)"] & A_1(2) \arrow[d, "f_1(2)"] \\
B_1(0) \arrow[r] & B_1(1) \arrow[r] & B_1(2) \\
 & s(b) & s(b)
\end{tikzcd}$$

3 Formal verification

Despite its compact formulation, the counterexample has a considerable complexity. For example, for each day the Kan conditions have to be verified, and for day 1 this amounts to 66 cases. Also, due to the identifications on day 2, one has to verify that equality is a congruence with respect to every function and relation, and in particular with respect to the filling operations.

In order to achieve the highest level of accuracy, we have formalized the complete countermodel in a fragment of first-order logic called coherent logic [3]. There are basically two things to verify: (1) the countermodel is indeed a Kripke model satisfying in every state the Kan conditions in Definition 2.2; (2) adding functions satisfying the conditions in Proposition 2.6 to the Kripke model leads to a contradiction.

We flatten all functions into functional relations and give all relations one extra parameter ranging over the poset $0 \leqslant 1 \leqslant 2$. Examples of axioms are now:

$$A_0(0, a) \quad B_0(0, b) \quad B_0(0, b') \quad G(0, w) \quad \mathit{edge}(0, w, a, b) \quad G(1, x) \quad \mathit{edge}(1, x, a, b') \quad \mathit{eq}_1(2, w, x)$$

$$\mathit{edge}(S, E, P_s, P_t) \iff d_1(S, E, P_s) \wedge d_0(S, E, P_t) \qquad \mathit{loop}(S, P, E) \iff s(S, P, E) \wedge \mathit{edge}(S, E, P, P)$$

Capitalized names (in term positions) denote variables, implicitly universally quantified. All predicates are monotonic in the state, including, for example, the flattened filling operations:

$$S_1 < S_2 \wedge \mathit{fill}_2(S_1, P, E_1, E_2, E_3) \Rightarrow \mathit{fill}_2(S_2, P, E_1, E_2, E_3)$$

The equality relations eq_0, eq_1 are congruences with respect to all predicates. An example of a Kan condition is:

$$A_0(S, P_1) \wedge A_0(S, P_2) \wedge B_0(S, P_3) \wedge A_1(S, E_1) \wedge G(S, E_2) \wedge edge(S, E_1, P_1, P_2), edge(S, E_2, P_1, P_3)$$
$$\Rightarrow \quad \exists E_3 \ (fill_2(S, P_1, E_1, E_2, E_3) \wedge G(S, E_3) \wedge edge(S, E_3, P_2, P_3))$$

Verification (1) is now essentially a model check, which can be performed in reasonable time.

Verification (2) is essentially a proof of the contradiction arising when one adds an arbitrary homotopy equivalence of the fibers, in the form of functional relations f_0, g_0, f_1, g_1, satisfying the conditions in Proposition 2.6, to the Kripke model. Examples of such axioms are:

$$A_0(S, P) \Rightarrow f_0(S, P, b) \vee f_0(S, P, b') \qquad B_0(S, P) \Rightarrow g_0(S, P, a) \vee g_0(S, P, a')$$

$$A_0(0, P_1) \wedge A_0(0, P_2) \wedge B_0(0, P_3) \wedge f_0(0, P_1, P_3) \wedge g_0(0, P_3, P_2) \Rightarrow eq_0(0, P_1, P_2)$$

$$f_0(S, P_1, P_2) \wedge s(S, P_1, E_1) \wedge s(S, P_2, E_2) \Rightarrow f_1(S, E_1, E_2)$$

The complete set of axioms for f_0, g_0, f_1, g_1 should express functionality, naturality with respect to s, d_0, d_1, and monotonicity in the state. Moreover, eq_0, eq_1 should be congruences with respect to f_0, g_0, f_1, g_1 as well. Once one has added these axioms, a contradiction is readily inferred. All relevant files can be found at `http://uf-ias-2012.wikispaces.com/Semi-simplicial+types` under Update 6/24.

We finish this section by expanding shortly on 'adding a homotopy equivalence to the Kripke model', as this touches the essence of the Kripke semantics of intuitionistic logic. What this phrase actually means is that one adds a homotopy equivalence in each state *requiring that these homotopy equivalences are monotonic in the state*. This monotonicity is crucial for intuitionistic provability. Even though there are fine homotopy equivalences on day 1 (e.g., with $f_0(a) = f_0(a') = b$, $f_1(s(a)) = f_1(s(a')) = f_1(v) = f_1(v') = s(b)$ etc.), these cannot be used in the Kripke model since f_0 is different on day 0.

4 Discussion and Conclusions

We have shown that a basic property of Kan fibrations, the homotopy equivalence of fibers over a connected base, cannot be proved constructively. It will be possible to obtain similar unprovability results for other properties of Kan fibrations.

We would like to say a few words about the correct interpretation of such unprovability results. In the first place, our unprovability result concerns the usual formulation of Kan simplicial set theory. It does not in any way preclude that it is possible to reformulate Kan simplicial set theory such that the basics can be proved constructively. One well-known technique is to include extra information in the definitions. In the case of a Kan graph one could, for example, mark the degenerate edges. As this amounts to postulating the decidability of degeneracy, the second proof of Proposition 2.5 would become constructive.

However, it is not clear whether this idea can be generalized. Let *marked simplicial sets* be simplicial sets in which the degenerate objects are marked. To be of interest, marked simplicial sets should be a category with sufficient extra structure to form a model of type theory. It is not clear how exponentials can be marked in a constructive way. We consider the possible constructive reformulation of Kan simplicial set theory as a challenging open problem.

Acknowledgement

The research for this paper has been carried out for a large part while the authors were member of the Institute for Advanced Study in Princeton, as part of the program *Univalent Foundations of Mathematics*. We are grateful for the generous support by the IAS and the Fund for Math.

We thank Peter Lumsdaine and Mike Shulman for the edges z, z' that were initially missing, and Bjørn Ian Dundås for Remark 2.4.

References

[1] C. Kapulkin and P. LeFanu Lumsdaine and V. Voevodsky, *The Simplicial Model of Univalent Foundations*, 2012. `http://arxiv.org/abs/1211.2851`

[2] B. Barras and T. Coquand and S. Huber, *A Generalization of Takeuti-Gandy Interpretation*, 2013. `http://uf-ias-2012.wikispaces.com/file/view/semi.pdf`

[3] M.A. Bezem and T. Coquand. *Automating Coherent Logic.* In G. Sutcliffe and A. Voronkov, editors, *Proceedings LPAR-12*, LNCS 3835, pages 246–260, Springer-Verlag, Berlin, 2005.

[4] S. Mac Lane, *Categories for the Working Mathematician*, Volume 5 of Graduate Texts in Mathematics, 2nd Edition, Springer, 2010.

[5] J.P. May, *Simplicial Objects in Algebraic Topology.* Chicago Lectures in Mathematics, 2nd Edition, University of Chicago Press, 1993.

[6] P.J. Goerss and J.F. Jardine, *Simplicial homotopy theory.* Volume 174 of Progress in Mathematics, Reprint of the 1999 edition, Springer, 2009.

[7] S. Kripke, *Semantical Analysis of Intuitionistic Logic I*, In: M. Dummett and J.N. Crossley, editors, Formal Systems and Recursive Functions, North-Holland Publishing Company, 1965.