Abstract—We present two finitary cut-free sequent calculi for
the modal μ-calculus. One is a variant of Kozen’s axiomatisation
in which cut is replaced by a strengthening of the induction rule
for greatest fixed point. The second calculus derives annotated
sequents in the style of Stirling’s ‘tableau proof system with
names’ (2014) and features a generalisation of the μ-regeneration
rule that allows discharging open assumptions. Soundness and
completeness for the two calculi is proved by establishing a
sequence of embeddings between proof systems, starting at Stirl-
ing’s tableau-proofs and ending at the original axiomatisation
of the μ-calculus due to Kozen. As a corollary we obtain a new,
constructive, proof of completeness for Kozen’s axiomatisation
which avoids the usual detour through automata and games.

I. INTRODUCTION

Modal μ-calculus is the extension of propositional modal
logic by two quantifiers, μ and ν, that range over fixed
points of propositional functions. More formally, the quantified
formulas μA(x) and νxA(x) are interpreted over directed
labelled graphs as, respectively, the least and greatest fixed
points of the (monotone) function formalising, at the semantic
level, the mapping x ↦ A(x). These quantifiers, combined
with modal language, permit the expression of a variety of
finite and infinite path quantification, giving the μ-calculus a
second order flavour.

Since its inception in the late 1960s, the modal μ-calculus
has become established as a central logic in computer science.
On the one hand, the calculus is sufficiently rich to encompass
many of the temporal logics used in system verification,
most prominently computational tree logic and propositional
dynamic logic. On the other hand, despite its expressive power,
the standard computational problems, such as validity and
model checking, remain decidable.

The earliest deductive system for the modal μ-calculus is a
Hilbert-style axiomatisation due to Kozen [1]. A natural for-
mulation of Kozen’s system as a (Tait-style) sequent calculus
expands the usual sequent rules for the modal logic K by fixed
point and induction inferences, and the logical rule of cut:

\[
\frac{\Gamma, A(x) \rightarrow (\sigma \times A(x)) \rightarrow \Gamma, \sigma \times A}{\Gamma, \sigma \times A(x)} \quad \frac{\Gamma, A(\Gamma) \rightarrow \Gamma, A}{\Gamma, A} \quad \frac{\Gamma, A, \Gamma, \Sigma}{\Gamma} \quad \text{cut}
\]

where σ ranges over the two quantifiers, \( \Sigma \) denotes the
negation of \( A \) (in negation normal form), and \( \Gamma \) denotes the
conjunction over negations of elements of \( \Gamma \).

The above proof system, henceforth denoted Koz, is known
to be both sound and complete for the modal μ-calculus.
Soundness was established by Kozen in [1], as was com-
pleteness for what is known as the aconjunctive fragment.
Completeness for the full calculus, however, was only proved
much later, by Walukiewicz [2].

Completeness was (and still is) a significant result. To
establish completeness, Walukiewicz isolates a class of for-
mulae, called disjunctive formulæ, that is, provably in Koz,
as expressive as the full language, and for which weak
completeness can be derived. The proof of equi-expressivity of
the disjunctive fragment and the full μ-calculus is extremely
involved, and depends heavily on automata and game theoretic
techniques, all operating on tableaux. For a valid formula A,
Walukiewicz’ proof comprises two parts combined by a cut: a
Koz-proof for the dual of an (exponentially larger) disjunctive
formula equivalent to \( A \), say \( B \), and a proof of \( B \lor A \).
In particular, the resulting proof bears little resemblance to
semantic validity arguments wherein syntactic constraints such
as disjunctiveness rarely materialise (cf. Example III.4).

As well as a desire to simplify the completeness proof for
Koz, it is natural to ask for a cut-free proof system for the
modal μ-calculus. An obvious candidate is the subsystem of
Koz without cut, which we denote Koz−. As there is no known
cut elimination algorithm for Koz, and Walukiewicz’ proof
makes essential use of the cut rule, completeness of Koz− re-
mains a significant open problem. Attention has thus shifted to
providing alternative cut-free proof systems for the μ-calculus,
such as the infinitary system \( K_\omega(\mu) \) of [3] and, more recently,
Stirling’s ‘tableau proof system with names’ [4]. Although cut-
free, these systems deviate from traditional axiomatisations
due to their use of infinitary or annotated inference rules.

A. Contribution

In this paper we provide two finitary cut-free sequent calculi
that are sound and complete for modal μ-calculus. The first
of these, denoted Koz−, is a strengthening of Koz− in which
the induction rule \( \text{ind} \) is replaced by the inference

\[
\frac{\Gamma, \nu xA(\Gamma) \rightarrow \Gamma, \nu xA(x) \rightarrow \Gamma}{\Gamma, \nu xA} \quad \text{ind}_x
\]

The new inference rule can be seen as combining the induction
rule in Koz with two general fixed-point principles:

\[
\nu xA(x, y) \leftrightarrow \nu xA(x, x) \quad \nu xA(x \lor x) \leftrightarrow \nu xA(x)
\]
the first of which is referred to as the “golden lemma of \( \mu \)-calculus” by Arnold and Niwinski [5].

The second finitary proof system we introduce, denoted Clo, discards the induction rule of Kozen\(^{-}\) in favour of a generalisation of the \( \nu \)-regeneration rule. The new inference has the form
\[
\frac{[\Gamma, \nu x A]}{} \\
\frac{\Gamma, A(\nu x A)}{\Gamma, \nu x A \nu \text{-clos}}
\]
where the sequent within the brackets is understood as an assumption of the proof which is discharged by \( \nu \text{-clos} \). Applications of the rule are subject to the condition that there is a thread from the formula \( A(\nu x A) \) to \( \nu x A \) (in the discharged sequent) that does not regenerate fixed point variables subsuming \( x \). This restriction is formalised by annotating formulae: each formula in a Clo-proof is labelled by a word from a finite set of names such as to record the regenerations of formulæ induced by the \( \mu \) and \( \nu \) inferences; the condition on applications of \( \nu \text{-clos} \) is then represented by the local requirement that the premise and discharged assumptions of the rule have identical annotations.

Soundness of Kozen\(^{-}\) is a consequence of the admissibility of \( \text{ind}_s \) in Kozen; soundness of Clo is obtained by embedding the system into Kozen\(^{-}\) via an interpretation of annotated formulæ as ‘plain’ formulæ. To derive completeness for both Kozen\(^{-}\) and Clo we make use of Stirling’s ‘tableau proof system with names’ from [4], and provide a direct embedding of Stirling’s system into Clo.

The two cut-free complete proof systems we introduce provide valuable insight into the proof theory of modal \( \mu \)-calculus. Foremost, a finitary and purely syntactic completeness proof for Kozen’s axiomatisation emerges which does not refer to arguments involving automata or games over infinite objects. Moreover, since the embeddings between the systems described above are algorithmic, and Stirling’s completeness proof in [4] can be represented as a finitary canonical model construction, we obtain a constructive proof of completeness. To illustrate the new completeness proof we provide a running example for the valid formula \( \mu x y \mathrm{B} \rightarrow \nu y \mu x \mathrm{B} \).

**B. Related Work**

Calculi based on circular, or cyclic, proofs are an important alternative to proofs using explicit induction, and have found applications in modal and first-order logics equipped with inductive and co-inductive properties. Notable uses in the framework of pure modal logic include temporal logics [6] and Gödel–Löb provability logic [7] where, in each case, a sequent calculus of circular proofs is defined which is sound, complete and cut-free. In [8], the equivalence of well-founded induction and circular reasoning is proved in the context of the first-order \( \mu \)-calculus with explicit approximations, albeit with the rule of cut present in both systems. Finally, cyclic proofs have been applied to first-order logic extended by inductively defined predicates [9], [10], and their connection to proofs by induction is explored further in [11], [12].

**C. Outline of Paper**

In the next section we fix the notation and definitions necessary for later work. Following this we briefly overview the relevant literature on proof systems for the modal \( \mu \)-calculus. The new proof systems we employ and the reductions between them form the content of Sections IV–VI. We begin by establishing regularity conditions that can be imposed on Stirling’s tableau-proofs. These are subsequently exploited in Section V where the calculus Clo is formally defined and an interpretation of Stirling’s system is given. In Section VI the proof system Clo is embedded into Kozen\(^{-}\) and it is shown that the latter is a subsystem of Kozen. We conclude with a discussion on the consequences and potential applications of our results.

Due to space considerations technical proofs are omitted from this abstract; full details are available in [13].

**II. Syntax and Semantics**

The set of \( \mu \)-calculus formulæ is given by the grammar

\[
A := p \mid \neg p \mid x \mid A \land A \mid A \lor A \mid [a]A \mid (a)A \mid \mu x A \mid \nu x A
\]

where \( p \) ranges over a set \( \text{Prop} \) of propositional constants, \( \alpha \) over a set \( \text{Act} \) of actions and \( x \) over a countably infinite set \( \text{Var} \) of variables. The sentential operators \( [a] \) and \( (a) \) are referred to as modalities and \( \sigma x \) for \( \sigma \in \{\mu, \nu\} \) is called a quantifier. An occurrence of a variable \( x \) in a formula \( A \) is bound if it is within the scope of a quantifier \( \sigma x \) and is free otherwise. A formula is closed if all variables are bound.

We define the following operations on formulæ. Set \( \perp := p \land \neg p \) and \( \top := p \lor \neg p \) for some fixed \( p \in \text{Prop} \) and define \( A \rightarrow B = \overline{A} \lor B \) where \( \overline{A} \) denotes the dual of \( A \) according to De Morgan duality with

\[
\overline{\mu x A} = \nu x \overline{A} \quad \overline{\nu x A} = \mu x \overline{A} \quad x = x \quad \overline{\text{p}} = p
\]

Note that variables are never negated, so for instance \( \mu x (y \lor [a]x) = \nu x (y \lor (a)x) \). On closed formulæ, however, duality agrees with classical negation (up to logical equivalence). As formulæ will always be taken up to \( \alpha \)-equivalence, we assume that renaming of variables is unnecessary. Thus when writing \( A(B) \) we implicitly mean that \( A(x) \) is a formula and no free variable of \( B \) becomes bound in the substitution \( A(B) \).

Semantics for the modal \( \mu \)-calculus is a direct extension of Kripke semantics for (multi-)modal logic incorporating variables and quantifiers. A frame, or labelled transition system, is a tuple \( \mathcal{K} = (K, R, \lambda) \) where \( R : \text{Act} \rightarrow K \times K \) and \( \lambda : \text{Prop} \rightarrow 2^K \). The set \( K \) is called the domain of \( \mathcal{K} \). A valuation (over \( \mathcal{K} \)) is a function \( v : \text{Var} \rightarrow 2^K \).

Given a frame \( \mathcal{K} = (K, R, \lambda) \), formula \( A \) and valuation \( v \) over \( \mathcal{K} \), define \( |A|_v^K \) by induction on \( A \):

\[
|x|_v^K = v(x) \quad |p|_v^K = \lambda(p) \quad |A \land B|_v^K = |A|_v^K \cap |B|_v^K \\
|[a]A|_v^K = \{ s \in K \mid \forall t \in K(s, t) \rightarrow R(a) \rightarrow t \in |A|_v^K \} \\
|\nu x A|_v^K = \bigcup\{ S \subseteq K \mid S \subseteq |A|_v^K | x \rightarrow_S \}
\]
where \( v[x \mapsto S] \) denotes the valuation \( v' \) given by \( v'(y) = v(y) \) for every \( y \in \text{Var} \setminus \{x\} \), and \( v'(x) = S \). The remaining connectives and quantifiers are defined dually. Since variables may appear only positively the semantics of formulæ, when treated as functions of free variables, are monotone: if \( v(x) \subseteq S \subseteq K \) then for every formulæ \( A \), \( |A|_v^K \subseteq |A|_{v \upharpoonright v \subseteq S}^{K} \). An easy exercise shows that the semantics is closed under unravelling fixed points, i.e.

\[
|\sigma \times A(x)|_v^K = |A(x)|_{v \upharpoonright v(\sigma \times A(x))} = |A(\sigma A)|_v^K.
\]  

(1)

In particular, for \( \sigma = \mu \) (resp. \( \sigma = \nu \)), the Knaster–Tarski theorem implies \( |\sigma A|_v^K \) is the unique least (resp. greatest) set \( S \) ordered under inclusion satisfying \( S = |A|_{v \subseteq S}^{K} \).

A formula \( A \) is satisfiable if there exists a frame \( \Gamma \) such that \( |A|_v^K \) is non-empty for some valuation \( v \), and is valid if \( \Gamma \) is not satisfiable, i.e. \( |A|_v^K \) is the domain of \( K \) for every frame \( \Gamma \) and valuation \( v \).

Fix a formulæ \( A \) and let \( \text{Var}_A \) denote the variable symbols (bound or free) occurring in \( A \). A induces a strict preorder \( <_A \) on \( \text{Var}_A \), called the subsumption ordering for \( A \), generated by \( x <_A y \) if \( x \) is free in some sub-formula \( \sigma B \) of \( A \). It is convenient to assume that formulæ come with compatible subsumption orderings. Indeed, there exists a preorder \( <_A \) on \( \text{Var} \) such that every finite preorder is embeddable in \( < \). Call a formulæ \( A \) well-named if \( <_A \) is a sub-structure of \( < \), i.e. \( \quad \forall \alpha \in \text{Var}_A \times \text{Var}_A \quad \alpha <_A \gamma \) if \( (\alpha, \gamma) \in \text{Var}_A \times \text{Var}_A \). A formula is \( \alpha \)-equivalent \( \Rightarrow \alpha \)-equivalent) to a well-named formulæ. Henceforth we assume all formulæ are well-named and write \( x \leq y \) if either \( x < y \) or \( x = y \). Observe that \( \bar{A} \) is well-named iff \( A \) is.

A thread is a sequence of formulæ \( \alpha = (A_i)_{i < N} \) (with \( N \leq \omega \)) such that for every \( n + 1 < N \), one of the following conditions hold:

- \( A_{n+1} = A_n \),
- \( A_{n+1} \) is an immediate sub-formula of \( A_n \),
- \( A_n = \sigma A(x) \) for some \( \sigma \) and \( x \), and \( A_{n+1} = A(A_n) \).

Let \( \alpha = (A_i)_{i \in \omega} \) be an infinite thread and fix \( \sigma \in \{\mu, \nu\} \). A variable \( x \) occurs infinitely often in \( \alpha \) as \( \sigma \) if for every \( i < \omega \) there exists \( j > i \) such that \( A_j = \sigma A \) for some \( A \). We call a thread \( \alpha \) \( \sigma \)-thread if there is a variable \( x \) that i) occurs infinitely often in \( \alpha \) as \( \sigma \) and ii) for all \( y \neq x \) that occur infinitely often in \( \alpha \) as \( \mu \) or \( \nu \), \( x \not< y \). Given a sequence \( \beta = (A_i)_{i \leq N} \) of sets of formulæ, a thread through \( \beta \) is any thread \( \alpha = (A_i)_{i < N} \) such that \( A_i \in \Gamma_i \) for every \( i < N \).

III. SEQUENT CALCULI FOR MODAL \( \mu \)-CALCULUS

We begin by outlining a basic sequent calculus which we call fixed point logic that serves as a basis for all the calculi we present in this paper. In the following, a \textit{sequent} is a finite set of closed formulæ. Sequents are denoted by \( \Gamma, \Delta \), etc., formulæ are identified with singleton sequents, and \( \Gamma, \Delta \) abbreviates \( \Gamma \cup \Delta \).

**Definition III.1.** Fixed point logic, denoted Fix, is the sequent calculus comprising the seven inference rules and axioms in

\begin{align*}
\text{Ax1: } & p, \bar{p} \quad \Gamma, B, C \quad \Gamma, B \quad \Gamma, C \quad \Gamma, B \wedge C \\
\text{Ax2: } & \nu x A, \mu x \bar{A} \quad \Gamma, A(B), A(C) \quad \Gamma, A(B \vee C) \\
\text{mod } & \Gamma, A \quad \Gamma, A(\sigma x A(x)) \quad \Gamma, \sigma x A \quad \sigma \Gamma, A \quad \text{weak}
\end{align*}

![Figure 1. Rules and axioms of fixed point logic, Fix.](image)

![Figure 2. Additional rules present in Koz.](image)

![Figure 3. Inference rules cut and strengthened induction.](image)

1This is a consequence of Fráissé’s Theorem from Model Theory, cf [14, Theorem 7.1.2].

The class of Fix-proofs is not particularly interesting. Although sound, Fix is not complete for the \( \mu \)-calculus since it is sound with respect to interpreting both quantifiers as any set satisfying the fixed point equation in (1). Complete proof systems can be obtained by extending Fix with further rules (and axioms) or relaxing the well-foundedness condition on proofs.

A. Kozen’s Axiomatization

A natural presentation of Kozen’s system as a sequent calculus is the extension of Fix by the rule \textit{ind} and axiom \textit{Ax2} in Figure 2 and the rule \textit{cut} in Figure 3. In the present paper we are concerned with variants of Kozen’s system without the cut rule, and in order to more easily accommodate these cut-free calculi we also include a generalisation of the disjunction rule, \textit{\forall d} in Figure 2, which although admissible in the presence of cut, is not obviously so in the systems lacking cut. Thus we define \textit{Koz} to be the extension of Fix by the three rules and axioms in Figure 2 and represent Kozen’s axiomatisation as the sequent calculus \textit{Koz} + \textit{cut} which we denote \textit{Koz}.

**Theorem III.2** (Kozen [1] and Walukiewicz [2]). Koz \textit{is sound and complete for the} \( \mu \)-\textit{calculus}.

As an example we provide a Koz-proof of the valid formula \( \mu x y B \rightarrow \nu y \mu x B \). First, however, we note the following generalisation of axiom \textit{Ax2}.

**Lemma III.3.** Let \( A(x_0, \ldots, x_k) \) be a formulæ with at most the designated variables free. If \( B_i \) and \( C_i \) are closed formulæ for each \( i \leq k \), then

\[ \{B_i, C_i\}_{i \leq k} \vdash_{\text{Koz}} \rightarrow \neg (B_0, \ldots, B_k), A(C_0, \ldots, C_k). \]

**Example III.4.** Fix a formulæ \( B(x, y) \) with at most \( x \) and \( y \) free. A Koz-proof of the valid sequent \( \{\nu x \mu y B, \nu y \mu x B\} \) is given in Figure 4. The proof is inspired by the semantic argument of [5, Proposition 1.3.4]. This example demonstrates
\[ \frac{\nu y B(x, \nu x A), \nu y B(x, C)} {\nu y B(x, C), \nu y B(x, C)} \mu \]

\[ \frac{\nu x B(x, C), \nu x B(x, C)} {\nu x B(x, C), \nu x B(x, C)} \mu \mu \]

\[ \frac{\nu x B(x, C), \nu x B(x, C)} {\nu x B(x, C), \nu x B(x, C)} \mu \mu \]

\[ \frac{\nu y B(x, C), \nu y B(x, C)} {\nu y B(x, C), \nu y B(x, C)} \mu \mu \]

\[ \frac{\nu y B(x, C), \nu y B(x, C)} {\nu y B(x, C), \nu y B(x, C)} \mu \mu \]

The non-triviality of generating Koz-proofs, which can be attributed to the impredicativity inherent in the induction rule.

### B. Tableau Proofs

The first deductive system for the \(\mu\)-calculus for which completeness was established is a system of ill-founded Fix derivations, referred to as tableaux, and is due to Niwinski and Walukiewicz [15]. Every sequent induces a class of trees obtained by applying the inference rules of Fix in a ‘bottom-up’ fashion, systematically decomposing formula into sequents of sub-formula. A cardinality argument shows that every infinite path must contain an infinitely repeating sequent and it so happens that validity can be characterised by a syntactic condition on the threads through these paths.

**Definition III.5.** A tableau for \(\Gamma\) is a (possibly infinite) Fix tree \(\pi\) with root \(\Gamma\) such that every infinite path through \(\pi\) contains an infinite \(\nu\)-thread.

**Theorem III.6** (Niwinski and Walukiewicz [15]). Let \(A\) be a guarded formula. \(A\) is valid iff there exists a tableau for \(A\).

Guardedness is the syntactic restriction on formulae requiring that for every sub-formula \(\sigma x B\) every occurrence of \(x\) in \(B\) is under the scope of a modality (in \(B\)). Every formula is equivalent to a guarded formula and this equivalence is provable in Koz [1]. However, the restriction to guardedness in Theorem III.6 turns out to be unnecessary:

**Theorem III.7.** A formula \(A\) of the \(\mu\)-calculus is valid iff there exists a tableau for \(A\).

Theorem III.7 is a corollary of the main result of Studer [16, Theorem 7.2] and also follows from Friedmann and Lange [17] on satisfiability tableaux for unguarded formula. A direct proof of the ‘left-to-right’ direction in the style of [15] can be obtained using instances of weak to eliminate the infinite \(\mu\)-threads that do not involve modalities (see for example [13, Theorem 3.6]). Although we do not use Theorem III.7 directly, the technique to accommodate unguarded formulæ in tableaux-based calculi is important for obtaining cut-free completeness where one cannot appeal to equivalences between guarded and unguarded formulæ.

### C. Semi-formal Systems

An alternative to the infinitely ‘long’ tableau proofs is to consider infinitely ‘wide’ proofs, where one or more inference rules take infinitely many premises. Such systems, known as semi-formal systems or infinitary calculi, have been widely studied in the context of first and second order theories of arithmetic.

Jäger, Kretz and Studer [3], drawing on this background, define a sound and complete cut-free proof system for \(\mu\)-calculus by adding an infinitary rule characterising the greatest fixpoint. For each \(n < \omega\), define a new ‘quantifier’ \(\nu^n\) by \(\nu^n x A = \top\) and \(\nu^{n+1} x A(x) = A(\nu^n x A)\). The \(\nu_\omega\) inference is the following infinitary rule, the premises of which are proofs of \(\Gamma, \nu^n x A\) for each \(n\).

\[ \frac{\nu^n x A \quad \Gamma, \nu^n x A \quad \Gamma, \nu^2 x A \quad \Gamma, \nu^3 x A \quad \cdots} {\Gamma, \nu x A} \]

Following [3] we call the extension of Fix by the above rule \(K_\omega(\mu)\). \(K_\omega(\mu)\)-proofs are infinite objects due to the \(\nu\)-branching of the \(\nu_\omega\) rule but well-foundedness ensures a \(K_\omega(\mu)\)-proof has no infinite paths.

Soundness for \(K_\omega(\mu)\) is an immediate consequence of the finite model property of the \(\mu\)-calculus: if the premises of the \(\nu_\omega\) rule are all valid sequents but the conclusion is not then there exists a finite frame \(K\) such that \(K \not \models \nu x A\) but \(K \models \nu^n x A\) for every \(n\), which yields a contradiction when \(n\) is at least the size of the domain of \(K\). Completeness is established via an infinitary version of the canonical model constructions for modal logics, and techniques from infinitary proof theory.

The finite model property implies that in each instantiation of \(\nu_\omega\) only finitely many of the premises are needed to deduce the validity of the conclusion. It does not by itself, however, provide any bound, but since the \(\mu\)-calculus satisfies also the small model property, a ‘finetisation’ of the \(\nu_\omega\) rule is possible. The resulting calculus, denoted \(K_{\omega,\omega}(\mu)\) in [3], is cut-free, sound and complete, and comprises only finitary inference rules. Nevertheless, the system is not entirely satisfactory as a finitary calculus due to inferences of arbitrarily high arity: in general, the \(\nu\)-rule deriving \(\Gamma, \nu x A\) has a number of premises exponential in the logical complexity of \(\Gamma\).

### D. Stirling Proofs

Stirling [4] introduces a ‘tableau proof system with names’ for the \(\mu\)-calculus that captures the infinite \(\nu\)-thread condition in tableaux within a finite tree. This is achieved by annotating formulæ and sequents with names for fixed point variables. The calculus derives from a dualisation of Jungteerapanich’s satisfiability tableaux presented in [18], [19]. We will be utilising a variant of Stirling’s system which we now present in detail.
For each variable symbol $x$, fix an infinite set $N_x$ of names for $x$. We assume $N_x \cap N_y = \emptyset$ if $x \neq y$ and use symbols $x, y, z$ (also with indices) as names for the formal variables $x, y$ and $z$ respectively. Let $N := \bigcup_{x \in \text{Var}} N_x$. Given a name $x \in N$, let $N_x$ denote the set $N_x$ such that $x \in N_x$. The subsumption ordering $\subseteq$ extends to names in the obvious way: for $x, y \in N$, $x \leq y$ ($x < y$) if $x \in N_y$ and $y \in N_x$, and $x \leq y$ (resp. $x < y$) for a set $M \subseteq N$ of names, $M^*$ is the set of finite words in $M$ including the empty word $\varepsilon$. For $a \in N^*$ and $x \in \text{Var}$ we write $a < x$ ($a \leq x$) if $a$ is a word in $\bigcup_{x \in N} N_x$ (resp. $\bigcup_{x \leq x} N_x$). Let $\subseteq$ denote the (reflexive) sub-word relation on $N^*$.

In Stirling’s proof system sequents are annotated by words from variable names. An annotation is a non-repeating word $a \in N^*$ weakly increasing in $<$: if $a = a_1 \ldots a_{k-1}$ then for all $0 \leq i < j < k$, $x_i \leq x_j$ and $x_i \neq x_j$. An annotated formula is a pair $(a, A)$, henceforth written as $A^a$, where $A$ is a closed formula and $a \in N^*$ is an annotation consisting of names for variables occurring in $A$. An annotated sequent is a finite set of closed annotated formulae $\{A^a_1, \ldots, A^a_n\}$ together with a finite word $a \in N^*$ without repetitions, called the control, written as $a \vdash A^a_1, \ldots, A^a_n$. Symbols $\Delta, \Pi$ range over finite sets of annotated formulae and identify a plain sequent $\{A_1, \ldots, A_k\}$ with the annotated sequent $\vdash A^a_1, \ldots, A^a_k$ in which the control and all annotations are the empty word.

Let fixed point logic with names, FixN, be the generalisation of Fix to annotated sequents with rules and axioms given in Figure 5. The notable restrictions on annotations are in the quantifier rules $\mu$ and $\nu$, wherein the annotation $b$ is required to comprise only names for $x$ and variables subsuming $x$, and in $\exp$, which permits expanding annotations by variable names restricted by the control. FixN will form the underlying calculus for the annotated proof systems in this paper. The first complete proof system extending FixN that we present is due to Colin Stirling [4].

**Definition III.8.** Let Stir denote the calculus extending FixN by the rule dis, and the rules $\nu_x$ and $\text{reset}_x$ for each variable name $x \in N$, listed in Figure 6. An open Stirling proof is a finite tree $\pi$ in $\text{Stir}$-inferences equipped with a function $l \mapsto l^c$ mapping each discharged assumption to the conclusion of the dis rule that discharged it such that:

1) for every annotated sequent $a \vdash \Gamma$ occurring in $\pi$ and every annotated formula $B^b \in \Gamma$, $b \subseteq a$;

2) for every discharged assumption $l$, there is a variable name $x$ appearing in the control of every node on the path from $l^c$ to $l$ inclusive, and an application of $\text{reset}_x$ on this path.

For each discharged assumption $l$ the node $l^c$ is called the companion of $l$ and $l$ a companion leaf of $l^c$. A companion node is a companion to some discharged assumption in $\pi$. The set of companion leaves to a node $m$ is denoted $c_r(m)$. If $\mathcal{A}$ is a set of annotated sequents, we write $\mathcal{A}, a \vdash \text{Stir} \Gamma$ if there is an open Stirling proof with conclusion $a \vdash \Gamma$ and open assumptions in $\mathcal{A}$. A Stirling proof, or Stir-proof, is a closed open Stirling proof, i.e. one with no open assumptions.

When presenting Stirling proofs, we illustrate the companion function $c$ by labelling discharged assumptions and the corresponding application of dis by symbols $\dagger$ and $\ddagger$ as in Figures 7 and 8. Note that in the presence of condition 1 in Definition III.8 the expansion rule $\exp$ simplifies to

$$\left(\forall i \leq k, b_i \mid a_0 = a_i\right) a_0 \vdash A^a_1, \ldots, A^a_{k_x} \\exp$$

The proof system Stir presented here differs slightly from [4] which is goal orientated with deterministic inference rules when read bottom-up. For instance, Stirling’s system contains neither of the rules $\text{weak}$ and $\exp$: the former appears in a restricted form, called thinning; the latter is instead incorporated directly into the other rules. It follows that each inference rule of Stirling’s system can be simulated in Stir by a combination of an inference in Stir and (possibly) an application of $\exp$ and weak without affecting the requirements on proofs in Definition III.8. Thus completeness for Stir is a corollary of the completeness proof in [4], using the argument in Theorem III.7 to generalise to unguarded formulæ.

**Theorem III.9.** Let $A$ be a closed well-named formula. If $A$ is valid then Stir $\vdash A$.

Stirling’s proof of Theorem III.9 proceeds by building a partial FixN proof, starting from the root, via a deterministic strategy for applying the inference rules. A cardinality argument ensures the construction terminates, yielding a proof if and only if the starting formula is valid.
Stirling also proves soundness for his system though we do not make use of it here. We will obtain an alternative proof of soundness for Stir (and, as a consequence, for the proof system of [4]) by embedding Stir into Koz.

We close the section with two examples: a generalisation of the law of excluded middle to annotated sequents and a Stir-proof of the valid (annotated) sequent system of [4]) by embedding Stir-inferences into the law of excluded middle to annotated sequents and a

\[ \text{Lemma IV.10. Let } a \in \mathbb{N}^* \text{ be a non-repeating word and } b, c \subseteq \text{ a annotations. Suppose } \nu x_0 \cdots \nu x_k A(x_0, \ldots, x_k) \text{ is a closed well-named formula. For each } i \leq k, \text{ let } b_i \text{ and } c_i \text{ be the restriction of } b \text{ and } c \text{ respectively to names in } \bigcup_{i \leq k} N_i. \text{ For all sequences of closed formulae } (B_i)_{i \leq k} \text{ and } (C_i)_{i \leq k}, \]

\[ \{ a \vdash B_i^b, C_i^c \}_{i \leq k}, a \vdash \text{Stir } \bigvee_{i \leq k} A(B_0, \ldots, B_k)^b, A(C_0, \ldots, C_k)^c. \]

\[ \text{Proof: The proof proceeds by structural induction on } A. \] The only non-trivial case is if } A \text{ is quantified. Suppose } A(x) = \nu y A(x, y). \text{ Without loss of generality, assume } x < y \text{ and } a, b, c \leq y. \text{ Let } y, y' \in N_y \text{ be names for } y \text{ not occurring in } a. \text{ The desired Stir-proof is given in Figure 7 where the omitted inferences are provided by the induction hypothesis.} \]

\[ \text{Example III.11. Starting with the annotated sequent } \varepsilon \vdash \nu x y \nu y B, \nu y x B^c. \text{ Stirling’s construction yields the proof } \pi_{\text{stir}} \text{ in Figure 7. In this proof, } C = \nu x y \nu y B, D = \nu y x B \text{ and the missing inferences are provided by the previous lemma.} \]

\[ \text{IV. UNFOLDING STIRLING PROOFS} \]

In this section we establish two important closure properties of Stir-proofs, called monotonicity and invariance, that will prove critical for embedding Stir into the proof system Clo introduced in the next section. We begin with some preliminary observations.

\[ \text{Definition IV.1 (Leaf invariants). Let } \pi \text{ be an open Stir-proof and } l \text{ a discharged assumption in } \pi. \text{ The invariant for } l, \text{ denoted } \text{invar}_c(l), \text{ is the shortest initial segment of the control at } l \text{ with the form } ax \text{ such that } x \text{ appears in the control of every node between } l^c \text{ and } l \text{ inclusive and there exists an application of } \text{reset}_c \text{ at some node between } l^c \text{ and } l. \text{ If no such word exists, we set } \text{invar}_c(l) = \varepsilon. \text{ If } \text{invar}_c(l) = ax, \text{ the conclusion to the first occurrence of } \text{reset}_c \text{ on the path from } l^c \text{ to } l \text{ will be referred to as the reset node for } l. \]

\[ \text{We may rephrase the definition of Stir-proofs in terms of leaf invariants:} \]

\[ \text{Lemma IV.2. A tree in Stir-inferences is a Stir-proof iff it satisfies condition 1 of Definition III.9 and every discharged leaf has non-trivial invariant.} \]

\[ \text{Lemma IV.3. Let } \pi \text{ be a Stir-proof with empty control at the conclusion.} \]

\[ \text{1) If } A^{axc} \text{ and } B^{bxd} \text{ are annotated formulae occurring in the same sequent in } \pi \text{ then } a = b, \]

\[ \text{2) If } l \text{ is a discharged assumption in } \pi \text{ then the control of every node between } l^c \text{ and } l \text{ is prefixed by } \text{invar}_c(l). \]

\[ \text{3) If } l_0 \text{ and } l_1 \text{ are two leaves of } \pi \text{ with the same invariant and associated the same companion node then their respective reset nodes are either the same or incomparable.} \]

As a discharged leaf and its associated companion are always labelled by the same annotated sequent, a Stirling proof can be unfolded by recursively replacing discharged assumptions by a fresh copy of their companion’s sub-proof. Depending on how one associates companions to newly created leaves, different unfoldings of a single proof can be constructed. The following definition makes precise the operation of unfolding a leaf and selecting companions for new leaves, and the subsequent lemma establishes that Stirling proofs are closed under all forms of unfolding.

\[ \text{Definition IV.4 (Unfolding Stir-proofs). Let } \pi \text{ be a Stirling proof and } M \text{ a non-empty set of discharged assumptions in } \pi \text{ whose companion nodes form a set of pairwise incomparable nodes. Let } O \text{ be a set of discharged leaves in } \pi \text{ with the property that for every } o \in O \text{ there exists } m \in M \text{ such that } m^c \leq o < m. \text{ We define the } O\text{-unfolding of } \pi \text{ at } M \text{ to be the tree } \pi' \text{ given by replacing each leaf } m \in M \text{ by a copy of the sub-proof of } \pi \text{ at } m^c \text{ in which assumptions from } O \text{ are left open in the sub-proof whenever possible. This condition is made precise by the definition of the companion function } e' \text{ of } \pi'. \text{ Let } e' : \pi' \rightarrow \pi \text{ be the function projecting } \pi' \text{ back to } \pi \text{ given by: if } n > m \text{ for some } m \in M \text{ then } \hat{n} \text{ is the node of } \pi \text{ from which } n \text{ was copied in the formation of } \pi'; \text{ if } n \in M \text{ then } \hat{n} = n^c; \text{ otherwise, } \hat{n} = n. \text{ Fix an arbitrary discharged} \]

\[ \begin{align*}
\text{Lemma III.10} & \\
\text{Proof:} & \\
\text{Figure 7. Two examples of Stir-proofs: on the left a proof of excluded middle for Lemma III.10; on the right the proof } \pi_{\text{stir}} \text{ of Example III.11.} &
\end{align*} \]
assumption \( l \) in \( \pi' \). If \( l \) is a leaf in \( \pi \) then \( l \) has the same companion as \( \bar{l} \) in \( \pi \), i.e. \( l^c = \bar{l}^c \). Otherwise, \( l > m \) for some \( m \in M \) and \( l' \) is defined according to the choice of \( O \) and position of \( l^c \):

- if \( \bar{l}^c < m^c \) then \( l' = \bar{l}^c \),
- if \( m^c \leq \bar{l}^c < m \) and \( l \in O \) then also \( l' = \bar{l}^c \),
- otherwise, \( m^c \leq l^c \) and \( l' \) is chosen to be the unique node \( m \leq o < l \) such that \( \bar{d} = l^c \).

An unfolding of \( \pi \) is the \( O \)-unfolding of \( \pi \) at \( M \) for some choice \( O \) and \( M \).

If \( \pi' \) is an unfolding of \( \pi \) then every node that is a leaf in both \( \pi \) and \( \pi' \) is assigned the same companion and invariant in both derivations. We may therefore always assume the function assigning companions to leaves in \( \pi' \) extends the companion function for \( \pi \) and thus uniformly denote by \( l^c \) the companion of a node \( l \) in either proof. Moreover, since \( \pi \) is an initial sub-tree of \( \pi' \) we may assume a single accessibility relation, \( <, \) on nodes in both trees.

**Example IV.5.** Consider the Stir-proof \( \pi_{str} \) in Figure 7. Let \( l_0 \) and \( l_1 \) name respectively the left and right leaf of \( \pi_{str} \). There are twelve unfoldings of \( \pi_{str} \), given by picking \( O, M \subseteq \{ l_0, l_1 \} \) (with \( M \) non-empty). The result with \( O = \{ l_0 \} \) and \( M = \{ l_1 \} \), for instance, is given in Figure 8 and is obtained by replacing the leaf \( l_1 \) by a copy of the sub-proof of \( \pi_{str} \) rooted at the conclusion to \( \text{dis}_1 \). The fresh copy of \( l_0 \) introduced above \( l_1 \) is assigned the original instance of \( \text{dis} \) as its companion.

With the same \( M \) but \( O = \{ l_0, l_1 \} \), the result is the same proof but in which all leaves are discharged by the instance of \( \text{dis}_1 \). With \( M = \{ l_0 \} \) and \( O = \{ l_1 \} \) the unfolding is defined in a symmetric manner, with \( l_0 \) unravalled and the fresh copy of \( l_1 \) inserted linked to the discharge rule near the root.

There are two extremes of unfolding \( \pi \) at a set of leaves \( M \), the \( 0 \)-unfolding and the \( L \)-unfolding where \( L \) is the set of all discharged assumptions \( o \) for which there exists \( m \in M \) such that \( m^c \leq o^c < m \). The former duplicates also the companion function, assigning fresh companions to new leaves if possible. The latter uses the most conservative choice of companion function possible, setting \( l^c = \bar{l}^c \) for every discharged assumption \( l \) satisfying \( \bar{l}^c < l \).

**Lemma IV.6.** Every unfolding of a Stir-proof is a Stir-proof.

**Proof:** Let \( \pi \) be a Stirling proof and \( \pi' \) be the \( O \)-unfolding of \( \pi \) at \( M \) for some choice of \( O \) and \( M \). It suffices to show that every discharged assumption in \( \pi' \) has non-trivial invariant. Without loss of generality assume \( M \subseteq c_\pi(m) \) for some companion node \( m \) in \( \pi \). Fix a node \( m' \in M \) and an arbitrary non-axiom leaf \( l > m' \) in \( \pi' \). Let \( l^c \) denote the companion of \( l \) in \( \pi' \) and \( l \) the projection of \( l \) into \( \pi \). If \( m' \leq l^c < l \) then \( \text{inv}_{\pi'}(l) = \text{inv}_{\pi}(l) \) by definition. Otherwise, \( l^c < m' \) and it follows that \( l^c \) and \( m' \) are comparable nodes, and \( l^c \) is the companion (in \( \pi \)) of the leaf \( l^c \). So \( m \leq l^c < m' \) or \( l^c < m < l \), whence Lemma IV.3.2 implies \( \text{inv}_{\pi}(m') \) and \( \text{inv}_{\pi}(l) \) are both prefixes of the control of one of \( l^c \) or \( m' \). Thus \( \text{inv}_{\pi}(m) \) and \( \text{inv}_{\pi}(l) \) are comparable and the shorter of the two is the invariant for \( l \) in \( \pi' \).

The next definition pin down two important regularity conditions on Stir-proofs, which are taken directly from [20].

**Definition IV.7** (Invariant and monotone Stir-proofs). A Stir-proof \( \pi \) is **invariant** if any two discharged assumptions with the same companion have the same invariant, and is **monotone** if for all pairs of assumptions \( m, n \) in \( \pi \) such that \( m^c < n^c < m \) the invariant of \( m \) is a prefix of the invariant of \( n \).

If \( \pi \) is an invariant Stir-proof, the function \( \text{inv}_{\pi} \) can be extended to companion nodes by setting \( \text{inv}_{\pi}(l^c) = \text{inv}_{\pi}(l) \) for every assumption \( l \). In a monotone invariant proof \( \pi \) the partial ordering of companion nodes by their invariants closely matches their position in the proof: for companion nodes \( m < n \) in \( \pi \), if \( n < l \) for some \( l \in c_{\pi}(m) \) then \( \text{inv}_{\pi}(m) \subseteq \text{inv}_{\pi}(n) \).

**Example IV.8.** The proofs constructed in Lemma III.10 are all monotone and invariant. The Stir-proof \( \pi_{str} \) in Example III.11 is also monotone but not invariant as the two assumption leaves associated to \( \text{dis}_1 \) have distinct invariants, namely \( x \) for \( l_0 \) and \( xy \) for \( l_1 \). For each \( i \in \{ 0, 1 \} \), let \( \pi_{str}^{1_i} \) be the \( \{ 1_i \} \)-unfolding of \( \pi_{str} \) at \( \{ 1_i \} \). \( \pi_{str}^{1_i} \) is neither monotone nor invariant, \( \pi_{str}^{1_0} \) is both monotone and invariant.

The next two results constitute the main contribution of the present section, that the restriction to monotone invariant Stir-proofs is also complete for \( \mu \)-calculus. This is established by repeatedly unfolding a given Stir-proof until first invariance
Lemma IV.9. For every Stirling proof there exists an invariant Stirling proof with the same conclusion.

Proof: Let $\pi$ be a Stirling proof. Let $m$ be a maximal node in $\pi$ that fails the invariance property, i.e., there exist leaves $n_1, n_2 \in c_{\pi}(m)$ such that $inv_{\pi}(n_1) \neq inv_{\pi}(n_2)$. Define an equivalence relation on $c_{\pi}(m)$ by setting $n \sim_{\pi} n'$ if $inv_{\pi}(n) = inv_{\pi}(n')$, and let $L_0, \ldots, L_k$ enumerate the equivalence classes such that $inv_{\pi}(n)$ is a proper prefix of $inv_{\pi}(n')$ if $n \in L_i, n' \in L_j$ and $i < j$. Such an enumeration is possible as the invariant of every leaf with companion $m$ is a prefix of the control at $m$. By assumption, $k > 0$. We prove there exists an unfolding $\pi'$ of $\pi$ such that no node in $\pi'$ strictly above $m$ fails the invariance property, and if $m$ fails the invariance property then the equivalence relation $\sim_{\pi'}$ on $c_{\pi'}(m)$ has fewer equivalence classes.

Set $O = \bigcup_{i<k} L_i$ and let $\pi'$ be the $O$-unfolding of $\pi$ at $L_k$. Notice that the invariant of every assumption in $O$ is a proper prefix of the invariant of nodes in $L_k$. Clearly, every companion node in $\pi'$ strictly above a node in $L_k$ fulfills the invariance property. Let $L'_0, \ldots, L'_k$ enumerate the $\sim_{\pi'}$-equivalence classes of $c_{\pi'}(m)$. By the proof of Lemma IV.6 and the choice of $O$, for every leaf $l \in L'_j, inv_{\pi'}(l) = inv_{\pi}(l)$, whence we conclude $k' < k$ and $L_i \subseteq \bigcup_{j<k} L'_j$ for $i < k$.

Theorem IV.10. Let $\Gamma$ be a sequent. Stir $\vdash \Gamma$ iff there exists a monotone invariant Stirling proof of $\Gamma$.

Proof: By the previous lemma it suffices to prove that every invariant Stirling-proof can be transformed into a monotone invariant Stir-proof. For each invariant proof $\pi$ define a relation $\prec_{\pi}$ between nodes in $\pi$ that witnesses any failure of monotonicity: $m \prec_{\pi} n$ if $m$ and $n$ are both companion nodes and there exists $m' \in c_{\pi}(m)$ such that i) $m < n < m'$, ii) $inv_{\pi}(n)$ is a prefix of $inv_{\pi}(m)$, and iii) if $inv_{\pi}(n) = inv_{\pi}(m)$ then the reset node of $m'$ is above $n$. The final condition, although not in the definition of monotonicity, is required to ensure termination of the unfolding process utilised below. Observe that $\prec_{\pi}$ is irreflexive and asymmetric, and that transitivity holds in some cases: $m \prec_{\pi} o \prec_{\pi} n$ and $n < m'$ for some $m' \in c_{\pi}(m)$ implies $m \prec_{\pi} n$. Furthermore, for companion nodes $m, n$ and $o$, the following closure property holds:

$$\text{if } m < o < n \text{ and } inv_{\pi}(o) \text{ is a prefix of } inv_{\pi}(n) \text{ then}$$

$$m \prec_{\pi} n \text{ implies } m \prec_{\pi} o \quad (2)$$

Let the rank of a companion node $m$, written $rk_{\prec_{\pi}}(m)$, be the length of the longest $\prec_{\pi}$-chain starting from $m$, i.e., the largest $r \geq 0$ for which there exist $0 \leq m_0 \prec_{\pi} m_1 \prec_{\pi} \cdots \prec_{\pi} m_r$ with $m = m_0$. If every node in an invariant proof $\pi$ has zero rank then $\pi$ is monotone. For a set $M$ of companion nodes in $\pi$ define $c_{\pi}(M) = \bigcup_{m \in M} c_{\pi}(m)$, and for $< \prec_{\pi}$ being $<$ or $\prec_{\pi}$, write $n < M (M < n)$ if there exists $m \in M$ such that $n < m$ (resp. $m < n$).

The proof of the theorem now proceeds by induction on the maximal rank of nodes and a subsidiary induction on the number of nodes of maximal rank. Assume $\pi$ is a closed invariant Stir-proof of $\Gamma$ and let $r + 1$ be the maximal rank among companion nodes in $\pi$. Fix a maximal companion node $m_0$ in $\pi$ with rank $r + 1$. We construct a sequence $\pi = \pi_0, \pi_1, \ldots, \pi_k$ of invariant proofs such that for each $i < k, \pi_{i+1}$ is an unfolding of $\pi_i$ at a set of nodes whose companions are at or above $m_0$.

The particular sequence of proofs we consider is defined as follows, where we write $\varsigma_i, c_i$ and $rk_i$ for $\varsigma_{\pi_i}, c_{\pi_i}$, and $rk_{\pi_i}$, respectively. Let $M_0 = \{m_0\}$. Suppose $\pi_i$ has been defined and $M_i$ is a set of companion nodes in $\pi_i$ with non-zero rank. Define $O_{i+1} = \bigcup\{c_i(o) \mid M_i \ni o\}$ and $M_{i+1} = \{m \in c_i(M_i) \mid \exists n(M_i \ni m < n)\}$ which denote, respectively, the companion leaves to nodes $c_i$-above $M_i$, and the set of companion leaves to a node in $M_i$ witnessing non-zero rank of elements of $M_i$. If $M_{i+1}$ is non-empty, set $\pi_{i+1}$ to be the $O_{i+1}$-unfolding of $\pi_i$ at $M_{i+1}$; otherwise we let $k = i$, which ends the sequence.

One can show, by induction on $i < k$, that

1) for every node $n$ in $\pi_{i+1}$, $rk_{i+1}(n) \leq rk_i(n);
2) for every node $n$ in $\pi_{i+1}$, if $rk_{i+1}(n) = r + 1$ and $n \geq m_0$ then $n \in M_{i+1};$
3) for every $m \in M_i, |M_{i+1} \cap c_i(m)| \leq |c_0(m_0)| - i$.

Property 3 implies the sequence of unfoldings terminates with $k \leq |c_0(m_0)|$. Since $M_{i+1}$ is empty iff $rk_i(m_0) = 0$ for every $m \in M_i, 1$ and $2$ imply that $\pi_k$ has fewer nodes of rank $r + 1$ than $\pi$.

Turning a Stir-proof into a monotone invariant proof via the above construction involves a substantial blow-up in size. Given a proof of height $h$ the construction yields an invariant proof of height bounded by $2^h$. Starting with an invariant proof of height $h$ and maximal rank $r + 1$, we obtain an invariant proof with rank $r$ and height also bounded by $2^h$. Since the transformation to invariant proofs does not increase the rank, the two bounds may be combined to deduce that the construction of monotone invariant proofs involves no worse than hyper-hyper-exponential increase in the height of Stir-proofs. A more efficient procedure may be possible by interleaving the steps for achieving invariance and monotonicity.

V. Circular Proofs with $\nu$-Closure

We now turn to the task of finding an alternative form for discharge rules that more succinctly describes the closure properties for the $\nu$-quantifier. This will be in the form of a fresh rule of inference, called $\nu$-closure, that generalises the annotated $\nu$ inference by permitting the discharging of assumptions. The $\nu$-closure rule will replace both the annotated $\nu$-rules and discharge rules of the preceding section, and although sequents in the new calculus are still annotated, these are in a much simpler form than Stir.

Definition V.1 (Proofs with $\nu$-closure). Let Clo be the proof system expanding FixN by the $\nu$-clo inference

\[
\Gamma, \nu A^x \vdash \nu A^x
\]

\[
\vdash \Gamma, \nu A^x \vdash
\]

\[
\vdash \Gamma \vdash, \nu A^x \vdash \nu A^x
\]

\[
(a \leq x \in N_x) \vdash \Gamma, (\nu A^x)^x \nu\text{-clo}_x
\]
with the restriction that $x$ does not appear in $\Gamma$. A Clo-proof is a finite tree in Clo-inferences in which all assumptions are discharged, all sequents have empty control and there is at most one use of $\nu$-clo, for each $x \in N$.

Recall that FixN, and hence also Clo, does not contain the rules $\nu_\sigma$ present in Stir. The rule is admissible in Clo however as it corresponds to an application of $\nu$-clo with no assumptions discharged.

Since controls in Clo-proofs are empty, the restriction on applications of $\exp$ in Clo reduces to only checking the subword relation, i.e. for the instance of $\exp$ given in Figure 5, the condition becomes $a_i \sqsubseteq b_i$ for every $i \leq k$. To simplify presentation, from now on we also identify a finite set $\Gamma$ of closed annotated formulæ with the annotated sequent $\epsilon \vdash \Gamma$.

**Lemma V.2.** For every formula $A(x_0, \ldots, x_k)$ with free variables among $x_0, \ldots, x_k$, all closed formulæ $B_i, C_i$ for $i \leq k$ and all annotations $b, c$ such that $b, c \leq x$ for all $\nu$-variables $x \in A$, 

$$\{B_0, \ldots, B_k\} \vdash \text{clo } \{A(B_0, \ldots, B_k)^b, A(C_0, \ldots, C_k)^c\}.$$

**Proof:** We deal with the quantifier case and assume $k = 0$, the other cases and the generalisation to $k > 0$ are straightforward. Let $A(y) = \nu x A_0(x, y), A_1(y) = A_0(A(y), y)$ and fix two closed formulæ $B$ and $C$ and annotations $b, c$ satisfying the hypothesis of the lemma. Arguing by induction, assume 

$$\{\overline{\text{A}(B)}^b, A(C)^{cz}\}, \{B^b, C^{cz}\} \vdash \text{clo } \overline{\text{A}(B)}^b, A_1(C)^{cz}$$

where $x$ is a fresh name for $x$. An application of $\mu$ yields 

$$\{\overline{\text{A}(B)}^b, A(C)^{cz}\}, \{B^b, C^{cz}\} \vdash \text{clo } \overline{\text{A}(B)}^b, A_1(C)^{cz}$$

and an application of $\nu$-clo at the root and $\exp$ at the remaining open assumptions yields $\{B^b, C^c\} \vdash \text{clo } \overline{\text{A}(B)}^b, A(C)^{c}$. \qed

One approach to proving completeness for Clo is to give a strategy for annotating tableaux which ensures infinite paths can be ‘collapsed’ to instances of the $\nu$-clo inference. In a sense, this is what we do, except that the tableaux and annotations are provided by a Stirling proof. Specifically, we prove that Clo-proofs can be directly extracted from unfoldings of monotone invariant Stir-proofs.

**Theorem V.3.** If $\text{Stir} \vdash \Gamma$ then $\text{Clo} \vdash \Gamma$.

**Proof:** The main difficulty in translating proofs in Stir into Clo is the dual role that variable names play in the former system. On the one hand, names control the invariants associated to assumptions via the $x \in N$ for which a reset rule is applied. On the other hand, names record (via annotations) the unravelling of $\nu$-quantifiers that are necessary for applications of reset. In Stir-proofs a given name may be utilised in both forms simultaneously: there may be assumptions $l_1, l_2$ with, say, $l_1^0 < l_2^0 < l_1$ and a name $x$ which occurs in the invariant for $l_1$ but is eliminated on the path from $l_2^0$ to $l_2$ by a reset rule. Although the restriction to monotone invariant Stir-proofs mitigates this particular problem, it will prove helpful to rephrase Stirling proofs within a framework that explicitly separates the two roles. For this we appeal to an intermediate calculus called Circ (‘circular proofs with discharge’) introduced in [20] for precisely this purpose. Circ-proofs correspond roughly to monotone invariant Stir-proofs in which the reset rule is applied exclusively at leaves; formally, the calculus replaces the dis, reset, and $\nu$-inferences of Stir by the inferences 

$$\begin{align*}
ax & \vdash \Gamma, \text{A}_{0, x, 0}^b, \ldots, \text{A}_{k, x, k}^b \\
\frac{\text{ax} \vdash \Gamma, \text{A}_{0, x, 0}^b, \ldots, \text{A}_{k, x, k}^b}{a \vdash \Gamma, \text{A}_{0, x, 0}^b, \ldots, \text{A}_{k, x, k}^b} \text{dis}_x \\
\frac{a \vdash \Gamma, \text{A}_{0, x, 0}^b, \ldots, \text{A}_{k, x, k}^b}{a \vdash \Gamma, \nu x \text{A}_{0}^b - \nu x}
\end{align*}$$

where $x, x_0, \ldots, x_k \in N_x, b \leq x$ and $x$ does not occur in $\Gamma$.

Every monotone invariant Stir-proof induces a Circ-proof with the same (annotated) conclusion [13], [20]. The construction involves successively removing instances of the reset rule, letting new annotations propagate through the proof to assumptions. Moreover, Circ-proofs can be easily transformed into Clo-proofs: inserting applications of $\exp$ at assumptions allows a Circ-proof to be repeatedly unfolded in the same manner as Stir-proofs (using the rules of the $\theta$-unfolding); a simple argument on the annotations and controls along the paths shows a bounded number of unfoldings suffices to find the instances of $\nu$-clo needed for a Clo-proof.

**Example V.4.** The construction described in Theorem V.3 may be applied to the monotone invariant Stir-proof $π_{\text{stir}}^\alpha$ in Figure 8 to obtain a Clo-proof of the sequent $\{\nu x \mu y B, \nu y \nu x B\}$. In this example, the transformation turns out to be quite simple. First, all instances of the reset rule that are not associated with an assumption are removed in favour of $\exp$. This only affects the instance of reset$_y$ near the root of the right-hand proof in Figure 8. As all reset rules now occur at leaves, the second step, which is to remove the remaining reset nodes and let the eliminated annotations propagate to assumptions, is trivial. Aside from fixing the controls, the result at this point is a proof in the calculus Circ described earlier.

The next step is to extract a Clo-proof from the Circ-proof. To achieve this, each discharged assumption must be unfolded until a $\nu$-thread becomes apparent on every path. In general, one must unfold each discharge rule by the number of its principal formulæ. For the example this means one unfolding of each assumption suffices. First the single companion leaf to the inference $\text{dis}_y$ is unfolded, and then the three companions of $\text{dis}_y$. After, on each of the (thirteen) paths to assumptions there will exist repetitions of the annotated $\nu$-rules (either $\nu_\sigma$ or $\nu_\sigma'$ depending on the path) connected by a (partial) $\nu$-thread; these rules become instances of $\nu$-clo. Finally, unwanted annotations can be removed, resulting in the proof $π_{\text{clo}}$ presented in Figure 9.

**VI. STRENGTHENED INDUCTION**

The final task is to embed the annotated proof system Clo into the plain sequent calculus $\text{Koz}_{\text{-}}$, the variant of Koz with the strengthened induction rule $\text{ind}_s$ of Figure 3, and to
embed the latter calculus within Koz. Note that Koz$^\sim$ may be viewed as an extension of Koz$^-$ since the induction rule ind is derivable in Koz$^\sim$ via the combination of inferences

\[ \Gamma, A(\Gamma) \quad \text{weak, } \land_d \]
\[ \Gamma, A(\Gamma \lor \nu x A(\Gamma \lor x)) \quad \nu \]
\[ \Gamma, \nu x A(\Gamma \lor x) \quad \text{inds} \]

**Theorem VI.1.** Let $\Gamma$ be a sequent. If $\text{Clo} \vdash \Gamma$ then Koz$^\sim \vdash \Gamma$.

Given a circular proof $\pi$ we will define a translation $* : A^a \to A^{ax}$ of annotated formulæ into plain formulæ such that the $*$-translation of each discharged assumption in $\pi$ is a valid sequent derivable in Koz$^\sim$, each inference rule of FixN is derivable in Fix + $\lor_d$, and each instance of $\nu$-closure is admissible in Koz$^\sim$. As we will see, the definition of $*$ depends only on the contexts in which variable names are utilised.

**Proof of Theorem VI.1.** Fix an assignment $\Phi = \{ A_x \mid x \in N \}$ of closed formulæ to variable names. We define the associated translation $*$ induced by $\Phi$ by structural induction. Variables and other atoms will be unchanged by $*$. As a result, given an annotated formula $A(x)^a$ and a plain formula $C$, we may write $B^{bs}(C)$ for the result of replacing all occurrences of $x$ by $C$ in $B^{bs}$. For atoms, modal and logical connectives define

\[
(B \lor C)^{ax} = B^{ax} \lor C^{ax} \quad (\langle a \rangle B)^{ax} = \langle a \rangle B^{ax} \\
(B \land C)^{ax} = B^{ax} \land C^{ax} \quad (\{ a \} B)^{ax} = \{ a \} B^{ax}
\]

For quantified formulæ, suppose $x \in \text{Var}$ and $a = bx_1 \cdots x_k$. $C$ is an annotation where $x_1, \ldots, x_k \in N_x$ and $b < x < c$. Then for $B = B(x)$ we set

\[
(\mu x B)^{ax} = \mu x B^{bs} \\
(\nu x B)^{ax} = A_{x_1} \lor \cdots \lor A_{x_k} \lor \nu x. B^{bs}(A_{x_1} \lor \cdots \lor A_{x_k} \lor x)
\]

For a set $\Delta$ of annotated formulæ, set $\Delta^* = \{ A^{ax} \mid A^a \in \Delta \}$.

Regardless of the choice of $\Phi$ (provided variable names are assigned closed formulæ), the interpretation described above translates the inferences of FixN to Koz$^\sim$-derivations. The only non-trivial case is the translations of $\nu$ and $\exp$ inferences: the former becomes an application of the $\nu$ fixed point rule in Fix (followed by weak and $\lor$) and the latter becomes a series of $\lor_d$ inferences. Since $*$ is also the identity on plain formulæ (i.e. $A^{ax} = A$ for every formula $A$), it remains to show how an appropriate choice of $\Phi$ allows the interpretation of discharged assumptions as derivable sequents and instances of $\nu$-clo as admissible rules in Koz$^\sim$.

Fix a closed Clo-proof $\pi$ with conclusion $\Gamma$, say. Let $N_\pi$ denote the set of variable names used in $\pi$. By recursion through the $\nu$-clo rules in $\pi$ we define the assignment $\Phi_\pi = \{ A_x \mid x \in N_\pi \}$: for $x \in N_\pi$, set $A_x = \Gamma^\sim_x$ where $\Delta$ is the set of side formulæ of the single instance of $\nu$-clo in $\pi$. This is well-defined since a) the definition of $B^{bs}$ depends only on the choice of $A_x$ for $x$ occurring in $b$; and b) the conclusion to an instance of $\nu$-clo in $\pi$ may only contain a name $y$ if $\nu$-clo$_y$ is closer to the root than $\nu$-clo$_x$.

We prove by induction through $\pi$ that for each annotated sequent $\Delta$ in $\pi$, Koz$^\sim \vdash \Delta^*$. As mentioned above, we need only concern ourselves with discharged assumptions and instances of $\nu$-clo. For each $x \in N_\pi$ let $\Gamma_x$ denote the set of side formulæ to the instance of $\nu$-clo$_x$ in $\pi$, so $A_x = \Gamma^\sim_x$. If $\Delta$ is a discharged assumption in $\pi$ then $\Delta = \Gamma_x \cup \{ \nu x A^{ax} \}$ for some $x \in N_x$, formula $A$ and $a \leq x$. and $(\nu x A)^{ax} = \Gamma^\sim_x \lor B$ for some closed formulæ $B$. Lemma III.3 implies Koz$^\sim \vdash \Gamma^\sim_x \lor B$, whence an application of weak and $\lor$ yields Koz$^\sim \vdash \Delta^*$. For interpreting $\nu$-clo, suppose $\Delta = \Gamma_x \cup \{ \nu x A^{ax} \}$ and, by the induction hypothesis, the sequent $\Gamma^\sim_x, A(\nu x A(x))^{ax}$ is derivable in Koz$^\sim$, where $x \in N_x$ and $a \leq x$. Pick $b < x$ and $x_1, \ldots, x_k \in N_x$ such that $a = bx_1 \cdots x_k$ and set

\[
B(x) = \Gamma^\sim_{x_1} \lor \cdots \lor \Gamma^\sim_{x_k} \lor x \quad A'(x) = A^{bs}(B(x)).
\]
Then \((\nu x A)^{\ast \ast} = B(\nu x A')\) and
\[
(\nu x A)^{\ast \ast} = B(\Gamma_x^v \lor x \cdot A'(\Gamma_x^v \lor x))
\]
\[
A(\nu x A)^{\ast \ast} = A'(\Gamma_x^v \lor x \cdot A'(\Gamma_x^v \lor x))
\]
and the following derivation interprets \(\nu\)-clo: 
\[
\frac{\Gamma_x^v, A'(\Gamma_x^v \lor x \cdot A'(\Gamma_x^v \lor x))}{\Gamma_x^v, \nu A'(\Gamma_x^v \lor x) \cdot \text{ind}_s} \quad \text{(3)}
\]
Thus \(\text{Koz}_\nu \vdash \Delta^v\) is derivable for each sequent \(\Delta\) occurring in \(\pi\). Since \(\Gamma^v = \Gamma\), we conclude \(\text{Koz}_\nu \vdash \Gamma\).

**Example VI.2.** Applying Theorem VI.1 to the proof \(\pi_{clo}\) of Example VI.4 yields a proof of the sequent \(\nu y \nu x \nu y B, \nu y \mu x B\) in \(\text{Koz}_\nu\). Let \(C = \nu y \nu x \nu y B\) and \(D = \nu y \mu x B\) as before. We have 
\[
\Gamma_x = \{\mu y B(x, D)\} \quad \Gamma_y = \{\mu y \nu \nu x \nu y B(C, y)\}
\]
\[
C^{\ast \ast} = \nu y \nu x B(x, D) \lor \nu y \nu x \nu y B(x, D) \lor x, y)
\]
\[
D^{\ast \ast} = \nu y B(C, y) \lor \nu y \mu x B(x, \nu y B(C, y) \lor y)
\]
Because \(x, y\) and \(y'\) are the only annotations occurring in \(\pi_{clo}\), the above cases provide the interpretation of all other formulæ: if \(F^v\) is any other annotated formula in the proof then \(F^{\ast \ast}\) is the result of replacing in \(F\), \(C^{\ast \ast}\) for \(C\), \(D^{\ast \ast}\) for \(D\) and \(D^{\ast \ast}\) for \(D\), depending whether \(c = x, y\) or \(y'\).

Under the interpretation, discharged assumptions become derivable sequents and instances of \(\nu\)-clo are replaced by the sequence of inferences in (3). For example, each of the three instances of \(\nu\)-clo 
\[
\frac{\mu y B(C^{\ast \ast}, y), \nu y \nu x B(x, D^{\ast \ast})}{\mu y B(C^{\ast \ast}, y), \nu y \nu x B(x, \nu y B(C^{\ast \ast}, y) \lor y)} \cdot \text{ind}_s
\]
The result is a \(\text{Koz}_\nu\)-proof of \(\nu y \nu x \nu y B, \nu y \mu x B\).

We complete the section with the embedding of \(\text{Koz}_\nu\) into \(\text{Koz}\). As the latter system is sound, this yields soundness of \(\text{Koz}_\nu\) and, in turn, \(\text{Clo}\) and \(\text{Stir}\).

**Theorem VI.3.** If \(\text{Koz}_\nu \vdash \Gamma\) then \(\text{Koz} \vdash \Gamma\).

**Proof:** We first show \(\text{Koz} \vdash \nu y \nu x \nu y B(y \lor x), \nu x B(x)\) for every formula \(B(x)\) with at most \(x\) free. Fix \(B(x)\) and let \(C = \nu y \nu x B(y \lor x)\). The \(\text{Koz}\)-proof 
\[
\frac{Ax2: \nu x B(C \lor z), \nu x B(C \lor x)}{C \lor \nu x B(C \lor x), C} \quad \text{\(\wedge\)}
\]
\[
\frac{B(C \lor v x B(C \lor x))}{\nu y \nu x B(y \lor x), B(C)} \cdot \text{ind}_{\mu, \nu}
\]
derives the desired sequent. The following derivation then establishes the admissibility of \(\text{ind}_s\) in \(\text{Koz}\):
\[
\frac{\Gamma, \nu x B(C \lor x) \cdot \text{ind}_{\mu, \nu}}{\Gamma, \nu x B(x)}
\]

**Example VI.4.** Example VI.2 describes a proof of \(\mu x y B \rightarrow \nu y \mu x B\) in \(\text{Koz}_\nu\) which can be readily transformed into \(\text{Koz}\) by replacing each instance of \(\text{ind}_s\) by \(\text{ind}\) and \(\text{cut}\). The resulting \(\text{Koz}\)-proof has essentially the same structure as the \(\text{Koz}_\nu\)-proof. It is interesting to note that the translation of the subproof \(\Delta\) in Figure 9 (which itself is already a proof of the desired sequent), has a similar structure to the semantically motivated \(\text{Koz}\)-proof in Example III.4 but with subtle differences in the application of induction.

**VII. Conclusion**

Combining the results of previous sections we obtain

**Theorem VII.1.** Let \(A\) be a closed well-named formula. The following are equivalent.

1) \(A\) is valid
2) \(\text{Stir} \vdash A\)
3) \(\text{Clo} \vdash A\)
4) \(\text{Koz}_\nu \vdash A\)
5) \(\text{Koz} \vdash A\).

Moreover, there exists a primitive recursive transformation between the proof systems in the direction 2 to 5.

Theorem VII.1 settles a number of questions regarding proof systems for \(\mu\)-calculus which we summarise below.

**A. Cut-free Completeness**

The system \(\text{Koz}_\nu\) is a natural variant of Kozen’s original axiomatisation wherein cut is dropped in favour of the strengthened induction rule
\[
\frac{\Gamma, \nu x A(C \lor x) \cdot \text{ind}_{\mu, \nu}}{\Gamma, \nu x A(x)}
\]

Specifically, the calculus derives only plain sequents and all inference rules have a fixed arity. As such, \(\text{Koz}_\nu\) marks (as far as the authors are aware) the first finitary sound and complete cut-free proof system for the modal \(\mu\)-calculus. Furthermore, the completeness of \(\text{Koz}_\nu\) reduces the long standing open problem of whether Kozen’s axiomatisation without cut is complete to whether the strengthened induction rule, \(\text{ind}_s\), is admissible in \(\text{Koz}_\nu\). As the proof of Theorem VI.3 demonstrates, this in turn equivalent to the admissibility of the inference
\[
\frac{\Gamma, \sigma y \sigma x A(y \lor x)}{\Gamma, \sigma A(x)}
\]
which permits contracting quantifiers of the same kind in simple contexts.

**B. Constructive Completeness**

The embeddings \(2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5\) of Theorem VII.1 close the gap between Stirling’s tableau proofs and Kozen’s axiomatisation, thereby answering the question posed by Stirling in [4] and providing a new proof of completeness for Kozen’s original axiomatisation. The argument is constructive,
and provides an algorithmic approach to obtaining proofs in Koz. The procedure involves a hyper-hyper-exponential blow-up in the size of the proof, which is due to the transformation of Stirling proofs to monotone invariant Stirling proofs (Lemma IV.10). Nevertheless, this is not always witnessed: in our running example of the valid formula $\mu vxyB \rightarrow v\forall xB$ (Examples III.11, V.4, VI.2 & VI.4) the transformation between proof systems remains feasible (in fact linear in $B$) and the resulting Koz-proof can be written down by hand.

C. Bridging Finitary and Infinitary Calculi

The finitary sequent calculus Clo is of independent interest as it is inter-translatable with the infinitary system $\mu(\mu)$ of [3]. In particular, via Clo, an embedding of $\nu(\mu)$ into Koz is possible.

The interpretation of Clo into $\nu(\mu)$ is described in [13]. Moreover, Studer [16] provides a transformation of $\nu(\mu)$-proofs into tableaux which, when combined with the completeness proof for Stir, yields an embedding of $\nu(\mu)$ in Stir. It would be interesting to see whether Studer’s interpretation can be adapted to a direct embedding of $\nu(\mu)$ in Clo as this would serve to further clarify the relation of the infinitary calculus to the systems Koz$^{-}$ and Koz.

D. Future Work

The cut-free finitary calculi introduced and the constructive completeness proof of Kozen’s axiomatisation obtained in this article open the door to a number of avenues of research. One such is to the problem of interpolation. It is known that uniform interpolation holds for the $\mu$-calculus [21]. The proof of this result is indirect and uses $\mu$-automata to show that the calculus interprets bisimulation quantifiers. Typically however, constructive proofs of Craig interpolation can be obtained from cut-free calculi and we expect this to be the case for the system Clo. Another application of interest is to the problem of cut elimination. While there are proof systems that are known to be cut-free complete, effective cut-elimination has only been established for small fragments of $\mu$-calculus such as propositional dynamic logic [22], common knowledge [23] and the one variable fragment [24]. It may prove more viable to study effective cut elimination in the context of the annotated proofs systems, such as Clo, due to their analytic form. Finally, it would be interesting to see whether the techniques of this paper can be adapted to yield a direct completeness proof (and cut-free calculus) for the axiomatisation of Venema’s coalgebraic fixed point logic of [25].

ACKNOWLEDGMENT

This research was supported by the programmes Oberwolfach Leibniz Fellows and Research in Pairs by the Mathematisches Forschungsinstitut Oberwolfach in 2015/16. The authors were also supported by, respectively, the Swedish Research Council grant no. 2016-03502 and the Knut and Alice Wallenberg Foundation. The authors wish to thank the anonymous referees for their comments and suggestions which have helped improve this contribution.

REFERENCES