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A Semantic Analysis of Structural Recursion

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Example 1: Addition of ordinal numbers in SML

```
datatype Nat = ...

datatype Ord = O
             | S of Ord
             | Lim of Nat -> Ord;

fun addord x O       = x
  | addord x (S y')  = S (addord x y')
  | addord x (Lim f) = Lim (fn z => addord x (f z))
```

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Example 2: A pattern matching proof in LEGO

```
$[leRefl: {T|ClTy}{t|ClTm T}{v:Val T Ts0 t}vLe v v];
[[S,T:ClTy][t:ClTm T][v:Val T Ts0 t][s:ClTm S]
 [w:Val S Ts0 s][R:Ty one][r:ClTm (UnfoldRec R)][x:ClV r]
    leRefl vUnit ==> leUnit
 || leRefl (vInl S v) ==> leInl S S (leRefl v)
 || leRefl (vInr S v) ==> leInr S S (leRefl v)
 || leRefl (vPair v w) ==> lePair (leRefl v) (leRefl w)
 || leRefl (vFold R x) ==> leFoldl R (leFoldr R (leRefl x))
];
```

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Goal: From *structural recursiveness* ...

$$\forall v.\ (\forall w < v.\ f(w) \Downarrow) \to f(v) \Downarrow$$

... infer termination

$$\forall v.\ f(v) \Downarrow$$

Outline:

1. Def. of the foetus system: types $\sigma \in \mathrm{Ty}(\vec{X})$, terms $t \in \mathrm{Tm}^\sigma[\Gamma]$
2. Def. of the evaluation strategy: syntactic values $v \in \mathrm{Val}^\sigma$, closures $\langle t; e\rangle \in \mathrm{Cl}^\sigma$, op. sem. $\Downarrow \subseteq \mathrm{Cl}^\sigma \times \mathrm{Val}^\sigma$
3. Def. of the semantics: "good" values $v \in [\![\sigma]\!]$
4. Def. of the structural ordering $<_{\sigma,\tau} \subseteq [\![\sigma]\!] \times [\![\tau]\!]$
5. Proof of the wellfoundedness $[\![\sigma]\!]$ w.r.t. $<$
6. Def. of the good terms $\mathrm{TM}^\sigma[\Gamma]$
7. Proof of the normalization: $\forall t \in \mathrm{TM}.\langle t; e\rangle \Downarrow$

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The foetus system

	Type	Terms / **Values**	Explanation
(Unit)	1	**()**	unit set
(Var)	X, Y, Z, ...	–	type variables
(Sum)	$\sigma + \tau$	**inl**, **inr**, case	disjoint sum
(Prod)	$\sigma \times \tau$	**(–,–)**, fst, snd	product
(Arr)	$\sigma \to \tau(\vec{X})$	λ, **rec**, – – (app)	function space
(Rec)	$\mathrm{Rec}\, X.\sigma(X)$	**fold**, unfold	recursive (fixed-point) type

$$\sigma(\mathrm{Rec}\, X.\sigma(X)) \quad \underset{\text{unfold}}{\overset{\text{fold}}{\rightleftarrows}} \quad \mathrm{Rec}\, X.\sigma(X)$$

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Example 3: Recursor for Nat in foetus

$$\mathbf{Nat} \equiv \mathrm{Rec}\, X.\, 1 + X$$

$$\mathbf{O} \equiv \mathrm{fold}(\mathrm{inl}())$$
$$\mathbf{S}(v) \equiv \mathrm{fold}(\mathrm{inr}(v))$$

$$\begin{aligned}\mathbf{R}^\sigma \equiv\ & \mathrm{rec}\, \mathrm{R}^{\sigma\to(\mathbf{Nat}\to\sigma\to\sigma)\to\mathbf{Nat}\to\sigma}.\ \lambda f_O^\sigma.\ \lambda f_S^{\mathbf{Nat}\to\sigma\to\sigma}.\ \lambda n^{\mathbf{Nat}}.\\ & \mathrm{case}(\mathrm{unfold}(n),\\ & \quad \_^1.\ f_O,\\ & \quad n'^{\mathbf{Nat}}.\ f_S\, n'\ (\mathrm{R}\, f_O\, f_S\, n'))\end{aligned}$$

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Example 4: addord in foetus

$$
\begin{aligned}
\mathbf{Ord} &\equiv \mathrm{Rec}\, X.(1+X)+(\mathbf{Nat}\to X) \\
\\
\mathbf{O} &\equiv \mathrm{fold}(\mathrm{inl}(\mathrm{inl}())) \\
\mathbf{S}(v) &\equiv \mathrm{fold}(\mathrm{inl}(\mathrm{inr}(v))) \\
\mathbf{Lim}(f) &\equiv \mathrm{fold}(\mathrm{inr}(f))
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{addOrd} \equiv\ & \mathrm{rec}\,\mathrm{addOrd}^{\mathbf{Ord}\to\mathbf{Ord}\to\mathbf{Ord}}.\ \lambda x^{\mathbf{Ord}}.\ \lambda y^{\mathbf{Ord}}.\ \mathrm{case}(\mathrm{unfold}(y), \\
& n^{1+\mathbf{Ord}}.\ \mathrm{case}(n, \\
& \qquad\qquad \_^{1}.\ x, \\
& \qquad\qquad y'^{\mathbf{Ord}}.\ \mathbf{S}(\mathrm{addOrd}\,x\,y')) \\
& f^{\mathbf{Nat}\to\mathbf{Ord}}.\ \mathbf{Lim}(\lambda z^{\mathbf{Nat}}.\ \mathrm{addOrd}\ x\ (f\,z)))
\end{aligned}
$$

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Operational semantics

Closures Cl^σ:

$\langle t^\sigma; e\rangle$	t term, e environment
$f^{\rho\to\sigma}@u^\rho$	f function value, u argument value

Evaluation relation $\Downarrow^\sigma \subseteq \mathrm{Cl}^\sigma \times \mathrm{Val}^\sigma$:

- big step
- call-by-value
- fixed evaluation strategy

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Semantics

Let $\vec{V} \subseteq \mathrm{Val}^{\vec{\tau}}$. Define $[\![\sigma(\vec{X})]\!]_{\vec{V}}$ inductively:

(Unit) $[\![1]\!] := \{()\}$

(Var) $[\![X_n]\!]_{\vec{V}} := V_n$

(Sum) $[\![(\sigma + \tau)(\vec{X})]\!]_{\vec{V}} := \{\mathrm{inl}(v) : v \in [\![\sigma(\vec{X})]\!]_{\vec{V}}\} \cup \{\mathrm{inr}(v) : v \in [\![\tau(\vec{X})]\!]_{\vec{V}}\}$

(Arr) $[\![\sigma \to \tau(\vec{X})]\!]_{\vec{V}} := \{f \in \mathrm{Val}^{\sigma \to \tau(\vec{\tau})} : \forall u \in [\![\sigma]\!].\ \exists v \in [\![\tau(\vec{X})]\!]_{\vec{V}}.\ f@u \Downarrow v\}$

(Rec) $[\![\mathrm{Rec}\, Y.\sigma(\vec{X}, Y)]\!]_{\vec{V}} := \mathrm{lfp}\,\mathcal{F}$, where we define $\mathcal{F}$ as

$$\begin{aligned}\mathcal{F} \quad : \quad & \mathcal{P}\left(\mathrm{Val}^{\mathrm{Rec}\, Y.\sigma(\vec{\tau}, Y)}\right) \to \mathcal{P}\left(\mathrm{Val}^{\mathrm{Rec}\, Y.\sigma(\vec{\tau}, Y)}\right) \\ & W \mapsto \mathrm{fold}\left([\![\sigma(\vec{X}, Y)]\!]_{\vec{V}, W}\right)\end{aligned}$$

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Fixed-point

Let $(\mathcal{U}, \subseteq)$ be a complete lattice, $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ an operator. The least fixed-point $F = \mathrm{lfp}\,\mathcal{F}$ is characterized by:

(ispfp) $\mathcal{F}(F) \subseteq F$

(ismpfp) $\forall A \in \mathcal{U}.\, \mathcal{F}(A) \subseteq A \to F \subseteq A$

Monotonicity

$$\forall \sigma(X).\ A \subseteq B \to [\![\sigma(X)]\!]_A \subseteq [\![\sigma(X)]\!]_B$$

Proof: Induction on σ:

(Arr) Show: For all $f \in [\![\sigma \to \tau(X)]\!]_A$ and $u \in [\![\sigma]\!]$ there is a $v \in [\![\tau(X)]\!]_B$ satisfying $f@u \Downarrow v$.

(Rec) Show: $[\![\mathrm{Rec}\, Z.\sigma(X, Z)]\!]_A \subseteq [\![\mathrm{Rec}\, Z.\sigma(X, Z)]\!]_B$.

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Substitution

$$[\![\sigma(X)]\!]_{[\![\tau]\!]} = [\![\sigma(\tau)]\!]$$

For $V \subseteq [\![\tau]\!]$ we get

$$[\![\sigma(X)]\!]_V \subseteq [\![\sigma(\tau)]\!]$$

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Example 5: All numerals are good

$$\mathrm{Val}^{\mathbf{Nat}} = \{(\mathrm{fold} \circ \mathrm{inr})^n(\mathrm{inl}()) : n \in \mathbb{N}\} \stackrel{!}{=} [\![\mathbf{Nat}]\!]$$

Show: $\mathrm{Val}^{\mathbf{Nat}}$ is smallest fixed-point of

$$\begin{aligned} \mathcal{F} \quad &: \quad \mathcal{P}\left(\mathrm{Val}^{\mathbf{Nat}}\right) \to \mathcal{P}\left(\mathrm{Val}^{\mathbf{Nat}}\right) \\ \mathcal{F}(W) \quad &:= \quad \{\mathrm{fold}(v) : v \in [\![1 + X]\!]_W\} \\ &= \quad \{\mathrm{fold}(\mathrm{inl}()), \mathrm{fold}(\mathrm{inr}(v)) : v \in W\} \end{aligned}$$

We must only show (ismpfp):

$$\text{(i)} \qquad \bigcup_{n\in\mathbb{N}} \mathcal{F}^n(\emptyset) \subseteq W$$

$$\text{(ii)} \qquad \mathrm{Val}^{\mathbf{Nat}} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{F}^n(\emptyset)$$

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Structural ordering

Idea: Values are trees, "<" is subtree relation.

Definition of $<_{\sigma,\tau}, \leq_{\sigma,\tau} \subseteq [\![\sigma]\!] \times [\![\tau]\!]$:

$$\text{(lerefl)}\quad \frac{}{v \leq_{\sigma,\sigma} v} \qquad\qquad \text{(lelt)}\quad \frac{w <_{\sigma,\tau} v}{w \leq_{\sigma,\tau} v}$$

$$\text{(ltinl)}\quad \frac{w \leq_{\rho,\sigma} v}{w <_{\rho,\sigma+\tau} \mathrm{inl}(v)} \qquad\qquad \text{(ltinr)}\quad \frac{w \leq_{\rho,\tau} v}{w <_{\rho,\sigma+\tau} \mathrm{inr}(v)}$$

$$\text{(ltarr)}\quad \frac{\exists v \in \mathrm{CoDom}(f). w <_{\rho,\tau} v}{w <_{\rho,\sigma\to\tau} f} \qquad\qquad \text{(learr)}\quad \frac{\exists v \in \mathrm{CoDom}(f). w \leq_{\rho,\tau} v}{w \leq_{\rho,\sigma\to\tau} f}$$

$$\text{(ltfold)}\quad \frac{w <_{\sigma,\tau(\mathrm{Rec}\,X.\tau(X))} v}{w <_{\sigma,\mathrm{Rec}\,X.\tau(X)} \mathrm{fold}(v)} \qquad\qquad \text{(lefold)}\quad \frac{w \leq_{\sigma,\tau(\mathrm{Rec}\,X.\tau(X))} v}{w \leq_{\sigma,\mathrm{Rec}\,X.\tau(X)} \mathrm{fold}(v)}$$

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Wellfoundedness

Def. of the accessible set $\mathrm{Acc}^\sigma \subseteq [\![\sigma]\!]$:

$$\text{(acc)}\quad \frac{\forall \tau, [\![\tau]\!] \ni w < v.\, w \in \mathrm{Acc}^\tau}{v \in \mathrm{Acc}^\tau}$$

All semantic values are accessible.

$$\forall \sigma(\vec{X}), \vec{\rho}.\ [\![\sigma(\vec{X})]\!]_{\mathrm{Acc}^{\vec{\rho}}} \subseteq \mathrm{Acc}^{\sigma(\vec{\rho})}$$

Proof by induction on generation of σ.

(Arr) Let $f \in [\![\sigma \to \tau(\vec{X})]\!]_{\mathrm{Acc}^{\vec{\rho}}}$. By monotonicity $f \in [\![\sigma \to \tau(\vec{\rho})]\!]$, and from the induction hypothesis $\mathrm{CoDom}(f) \subseteq [\![\tau(\vec{X})]\!]_{\mathrm{Acc}^{\vec{\rho}}} \subseteq \mathrm{Acc}^{\tau(\vec{\rho})}$ we infer $f \in \mathrm{Acc}^{\sigma\to\tau(\vec{\rho})}$ (we need a small lemma).

(Rec) Show $[\![\mathrm{Rec}\,\sigma(\vec{X}, Y)]\!]_{\mathrm{Acc}^{\vec{\rho}}} \subseteq \mathrm{Acc}^{\mathrm{Rec}\,Y.\sigma(\vec{\rho},Y)}$. We use the induction hypothesis $[\![\sigma(\vec{X}, Y)]\!]_{\mathrm{Acc}^{\vec{\rho}},\mathrm{Acc}^{\mathrm{Rec}\,Y.\sigma(\vec{\rho},Y)}} \subseteq \mathrm{Acc}^{\sigma(\vec{\rho},\mathrm{Rec}\,Y.\sigma(\vec{\rho},Y))}$ and the fixed-point properties.

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Structural recursive terms

$$\begin{aligned}\mathrm{SR}^{\sigma\to\tau}[\Gamma] \quad := \quad & \{\mathrm{rec}\, g.t \in \mathrm{Tm}^{\sigma\to\tau}[\Gamma] : \forall e \in [\![\Gamma]\!], v \in [\![\sigma]\!]. \\ & (\forall [\![\sigma]\!] \ni w < v. \langle \mathrm{rec}\, g.t; e\rangle @ w \Downarrow) \to \langle \mathrm{rec}\, g.t; e\rangle @ v \Downarrow\}\end{aligned}$$

Induction principle for wellfounded sets:

$$(\mathrm{accind}) \qquad \frac{\forall v \in [\![\sigma]\!].\, (\forall [\![\sigma]\!] \ni w < v.\, P(w)) \to P(v)}{\forall v \in \mathrm{Acc}^{\sigma}.\, P(v)}$$

All structural recursive terms are good:

$$t \in \mathrm{SR}^{\sigma\to\tau}[\Gamma],\ e \in [\![\Gamma]\!] \to \langle t; e\rangle \in [\![\sigma \to \tau]\!]$$

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Example 6: addord is structural recursive

Recursive calls:

$\ldots \mathbf{S}(\mathrm{addOrd}\, x'\, y) \ldots$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v' \le v'}\,\mathrm{lerefl}}{v' < \mathrm{inr}(v')}\,\mathrm{ltinr}}{v' \le \mathrm{inr}(v')}\,\mathrm{lelt}}{v' < \mathrm{inl}(\mathrm{inr}(v'))}\,\mathrm{ltinl}}{v' < \mathbf{S}(v') \equiv \mathrm{fold}(\mathrm{inl}(\mathrm{inr}(v')))}\,\mathrm{ltfold}$$

$\ldots \mathbf{Lim}(\lambda z^{\mathbf{Nat}}.\ \mathrm{addOrd}\ (f\ z)\ y) \ldots$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{w \le w}\,\mathrm{lerefl}}{\exists v' \in \mathrm{CoDom}(f).\, w \le v'}\,\exists\mathcal{I}}{w \le f}\,\mathrm{learr}}{w < \mathrm{inr}(f)}\,\mathrm{ltinr}}{w < \mathbf{Lim}(f) \equiv \mathrm{fold}(\mathrm{inr}(f))}\,\mathrm{ltfold}$$

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Normalization

Define the good terms $\mathrm{TM}^\sigma[\Gamma] \subset \mathrm{Tm}^\sigma[\Gamma]$ inductively in the same way as Tm with the exception

$$\text{(REC)} \qquad \frac{t \in \mathrm{TM}^{\sigma\to\tau}[\Gamma, g^{\sigma\to\tau}] \quad \mathrm{rec}\, g.t \in \mathrm{SR}^{\sigma\to\tau}[\Gamma]}{\mathrm{rec}\, g.t \in \mathrm{TM}^{\sigma\to\tau}[\Gamma]}$$

Show normalization

$$\forall \sigma, \Gamma, t \in \mathrm{TM}^\sigma[\Gamma], e \in [\![\Gamma]\!].\ \langle t; e\rangle \Downarrow$$

by induction on t using the operational sematics.

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Extensions and open questions

- positive types (?)
- polymorphic types $\surd$
- dependent types
- coinductive types