Coinduction in Agda using Copatterns and Sized Types

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Copatterns

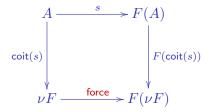
- Copatterns: "invented" to integrate sized coinductive types with pattern matching.
- Inspired by coalgebraic approach to coinduction (Anton Setzer).
- "Solved" the subject reduction problem of dependent matching on codata.

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- Operational semantics is WYSIWYG.
- Implemented in Agda 2.3.4 (forthcoming).

Coalgebras

• Copatterns = pattern matching for coalgebras.



• Computation: Only unfold infinite object on demand.

force $(\operatorname{coit} s a) = F(\operatorname{coit} s)(s a)$

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Copatterns: Syntax

• Elimination contexts (spines):

$$E ::= \bullet \quad \text{head} \\ | E t \quad \text{application} \\ | \pi E \quad \text{projection} \end{cases}$$

• Copatterns = pattern matching elimnation contexts.

• Rule Q[f] = t fires if copattern matches elimination context.

$$\frac{E = Q\sigma}{E[f] \longrightarrow t\sigma}$$

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Example: De Bruijn Lambda Terms and Values

```
data Tm (n : \mathbb{N}): Set where
var : (x : Fin n) \rightarrow Tm n
abs : (t : Tm (suc n)) \rightarrow Tm n
app : (r s : Tm n) \rightarrow Tm n
```

```
mutual

record Val : Set where

constructor clos

field \{n\} : \mathbb{N}

body : Tm (suc n)

env : Env n

Env = Vec Val
```

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Running Example: Naive Call-By-Value Interpreter

Evaluator (draft).

```
mutual

\begin{bmatrix} \_ \end{bmatrix}\_ : \forall \{n\} \to \mathsf{Tm} \ n \to \mathsf{Env} \ n \to \mathsf{Val}
\begin{bmatrix} var \ x \ \end{bmatrix} \ \rho = \mathsf{lookup} \ x \ \rho
\begin{bmatrix} abs \ t \ \end{bmatrix} \ \rho = \mathsf{clos} \ t \ \rho
\begin{bmatrix} app \ r \ s \ \end{bmatrix} \ \rho = \mathsf{apply} \ (\llbracket \ r \ \rrbracket \ \rho) \ (\llbracket \ s \ \rrbracket \ \rho)
apply : \mathsf{Val} \to \mathsf{Val} \to \mathsf{Val}
apply (\mathsf{clos} \ t \ \rho) \ v = \llbracket \ t \ \rrbracket \ (v :: \rho)
```

Of course, termination check fails!

The Coinductive Delay Monad

```
CoInductive Delay (A : Type) : Type :=
| return (a : A)
  later (a? : Delay A).
  mutual
     data Delay (A : Set) : Set where
        return : (a : A) \rightarrow \mathsf{Delay} A
        later : (a' : \mathsf{Delay}' A) \to \mathsf{Delay} A
     record Delay' (A : Set) : Set where
        coinductive
        constructor delay
        field force : Delay A
  open Delay' public
```

The Coinductive Delay Monad (Ctd.)

Nonterminating computation.

forever : $\forall \{A\} \rightarrow \mathsf{Delay'} A$ force forever = later forever

Monad instance.

mutual

$$_ \gg = _: \forall \{A \ B\} \rightarrow \mathsf{Delay} \ A \rightarrow (A \rightarrow \mathsf{Delay} \ B) \rightarrow \mathsf{Delay} \ B$$

return $a \ \gg = k = k \ a$
later $a' \ \gg = k = \text{later} (a' \gg = 'k)$

 $_ ='_: \forall \{A \ B\} \rightarrow \mathsf{Delay'} \ A \rightarrow (A \rightarrow \mathsf{Delay} \ B) \rightarrow \mathsf{Delay'} \ B$ force $(a' = k) = \mathsf{force} \ a' = k$

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Evaluation In The Delay Monad

Monadic evaluator.

$$\begin{array}{c} \llbracket _ \rrbracket _ : \forall \{n\} \to \mathsf{Tm} \ n \to \mathsf{Env} \ n \to \mathsf{Delay} \ \mathsf{Va} \\ \llbracket \ \mathsf{var} \ x & \rrbracket \ \rho = \mathsf{return} \ (\mathsf{lookup} \ x \ \rho) \\ \llbracket \ \mathsf{abs} \ t & \rrbracket \ \rho = \mathsf{return} \ (\mathsf{clos} \ t \ \rho) \\ \llbracket \ \mathsf{app} \ r \ s & \rrbracket \ \rho = \mathsf{apply} \ (\llbracket \ r \ \rrbracket \ \rho) \ (\llbracket \ s \ \rrbracket \ \rho) \end{array}$$

$$\begin{array}{l} \mathsf{apply}: \ \mathsf{Delay} \ \mathsf{Val} \to \mathsf{Delay} \ \mathsf{Val} \to \mathsf{Delay} \ \mathsf{Val} \\ \mathsf{apply} \ u^{\it Q} \ v^{\it Q} = \ u^{\it Q} \ \gg = \lambda \ u \to \\ v^{\it Q} \ \gg = \lambda \ v \to \\ \mathsf{later} \ \mathsf{(apply'} \ u \ v) \end{array}$$

apply' : Val
$$\rightarrow$$
 Val \rightarrow Delay' Val
force (apply' (clos $t \rho$) v) = $\llbracket t \rrbracket (v :: \rho)$

Not guarded by constructors!

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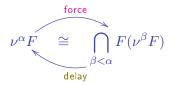
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Sized Coinductive Types

- Track guardedness in the type system (Hughes Pareto Sabry 1996).
- Size = iteration stage towards greatest fixed point.
- Deflationary iteration (F need not be monotone).



- $\nu^0 F = \top$ universe of terms / terminal object.
- Contravariant subtyping $\nu^{\alpha}F \leq \nu^{\beta}F$ for $\alpha \geq \beta$.
- Stationary point $\nu^{\infty+1}F = \nu^{\infty}F$ reached for some ordinal ∞ .

Sized Coinductive Delay Monad

```
\begin{array}{l} \mbox{mutual} \\ \mbox{data Delay } \{i: \mbox{Size}\} \ (A: \mbox{Set}) : \mbox{Set where} \\ \mbox{return} : \ (a : A) & \rightarrow \mbox{Delay } \{i\} \ A \\ \mbox{later} : \ (a': \mbox{Delay'} \ \{i\} \ A) & \rightarrow \mbox{Delay } \{i\} \ A \\ \mbox{record Delay'} \ \{i: \mbox{Size}\} \ (A: \mbox{Set}) : \mbox{Set where} \\ \mbox{coinductive} \\ \mbox{constructor delay} \\ \mbox{field} & \mbox{force} : \ \forall \{j: \mbox{Size} < i\} \rightarrow \mbox{Delay } \{j\} \ A \\ \mbox{open Delay' public} \end{array}
```

- Size = depth = how often can we force?
- Not to be confused with "number of laters"?

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Sized Coinductive Delay Monad (II)

Corecursion = induction on depth.

forever : $\forall \{i \ A\} \rightarrow \mathsf{Delay'} \ \{i\} \ A$ force (forever $\{i\}$) $\{j\} = \mathsf{later} \ (\mathsf{forever} \ \{j\})$

Since j < i, the recursive call forever $\{j\}$ is justified.

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Sized Coinductive Delay Monad (III)

Monadic bind preserves depth.

mutual

$$_ = _ : \forall \{i \ A \ B\} \rightarrow \\ \mathsf{Delay} \ \{i\} \ A \rightarrow (A \rightarrow \mathsf{Delay} \ \{i\} \ B) \rightarrow \mathsf{Delay} \ \{i\} \ B \\ \mathsf{return} \ a \ \gg = k = k \ a \\ \mathsf{later} \ a' \ \gg = k = \mathsf{later} \ (a' \ \gg = ' k)$$

$$\begin{array}{l} _ \circledast = '_ & : \ \forall \{i \ A \ B\} \rightarrow \\ & \mathsf{Delay'} \ \{i\} \ A \rightarrow (A \rightarrow \mathsf{Delay} \ \{i\} \ B) \rightarrow \mathsf{Delay'} \ \{i\} \ B \\ \texttt{force} \ (a' \ \gg = ' \ k) = \texttt{force} \ a' \ \gg = k \end{array}$$

Depth of $a? \gg = k$ is at least minimum of depths of a? and k a.

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Sized Corecursive Evaluator

Add sizes to type signatures.

$$\begin{array}{c} \llbracket _ \rrbracket _ : \forall \{i \ n\} \to \mathsf{Tm} \ n \to \mathsf{Env} \ n \to \mathsf{Delay} \ \{i\} \ \mathsf{Val} \\ \llbracket \ \mathsf{var} \ x \ \ \rrbracket \ \rho = \mathsf{return} \ (\mathsf{lookup} \ x \ \rho) \\ \llbracket \ \mathsf{abs} \ t \ \ \rrbracket \ \rho = \mathsf{return} \ (\mathsf{clos} \ t \ \rho) \\ \llbracket \ \mathsf{app} \ r \ s \ \ \rrbracket \ \rho = \mathsf{apply} \ (\llbracket \ r \ \rrbracket \ \rho) \ (\llbracket \ s \ \rrbracket \ \rho)$$

$$\begin{array}{l} \mathsf{apply}: \forall \{i\} \rightarrow \mathsf{Delay} \ \{i\} \ \mathsf{Val} \rightarrow \mathsf{Val} \rightarrow \mathsf{Val} \ \mathsf{v}^{?} \ \ \mathsf{v} = \lambda \ u \ \rightarrow \mathsf{v}^{?} \ \ \mathsf{v} = \lambda \ v \ \rightarrow \mathsf{later} \ \mathsf{(apply'} \ u \ v) \end{array}$$

$$\begin{array}{l} \mathsf{apply}' : \forall \{i\} \rightarrow \mathsf{Val} \rightarrow \mathsf{Val} \rightarrow \mathsf{Delay}' \; \{i\} \; \mathsf{Val} \\ \mathsf{force} \; (\mathsf{apply}' \; (\mathsf{clos} \; t \; \rho) \; v) = \llbracket \; t \; \rrbracket \; (v :: \; \rho) \end{array}$$

Termination checker is happy!

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Example: Fibonacci Stream

```
record S i A : Set where
coinductive
field head : A
tail : \forall \{j : \text{Size} < i\} \rightarrow \text{S} j A
open S
```

```
\begin{array}{l} \mathsf{zipWith} : \forall \{i \ A \ B \ C\} \to (A \to B \to C) \to \mathsf{S} \ i \ A \to \mathsf{S} \ i \ B \to \mathsf{S} \ i \ C \\ \mathsf{head} \ (\mathsf{zipWith} \ f \ s \ t) = f \ (\mathsf{head} \ s) \ (\mathsf{head} \ t) \\ \mathsf{tail} \ (\mathsf{zipWith} \ f \ s \ t) = \mathsf{zipWith} \ f \ (\mathsf{tail} \ s) \ (\mathsf{tail} \ t) \end{array}
```

```
\begin{array}{l} \mbox{fib} : \forall \{i\} \rightarrow {\sf S} \ i \ \mathbb{N} \\ ( & (\mbox{head} \ \mbox{fib})) = 0 \\ (\mbox{head} \ \mbox{(tail} \ \ \mbox{fib})) = 1 \\ (\mbox{tail} \ \ \mbox{tail} \ \ \mbox{fib})) = zip \mbox{With} \ + \ \mbox{fib} \ \mbox{(tail} \ \mbox{fib}) \end{array}
```

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Conclusions

- Type-based termination allows for natural corecursive programming.
 - Well-founded induction works around termination checker.
 - Nice work-around productivity checker?! (Danielsson 2010: DSLs, invasive.)

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- Compatible with Isomorphism-as-Equality (HoTT).
- Available now!
- Not completely for free; user needs to refine type signatures.
- Size constraint solver could be more powerful.

Related Work

- 1980/90s: Mendler, Pareto, Amadio, Giménez.
- 2000s: Barthe, Uustalu, Blanqui, Riba, Roux, Gregoire, ...
- Sacchini: LICS 2013, Coq.
- Coalgebraic types: Hagino (1987), Cockett: Charity (1992).
- Acknowledgment: Invitations to McGill (Pientka), Tallinn (Uustalu).